Corrections to *J. of the Ast. Sc.* paper [1]

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This note describes corrections to the [1] paper. The correct form of Eq. 41 in the paper is as follows:

$$H^* D^{-1} H = \Upsilon - \text{diag}\left\{\mathcal{E}_\psi^* \Upsilon \mathcal{E}_\psi\right\} \tag{41}$$

The paper is missing the *diag* term. The solution Υ is a block-diagonal matrix whose block-diagonal elements are computed recursively as described in the algorithm below Eq (42) in the paper.

Eq (42) in **the paper should include the additional R matrix** as follows:

$$\Omega = \Upsilon + \tilde{\psi}^* \Upsilon + \Upsilon \tilde{\psi} + R \tag{42}$$

where the additional matrix R is given by

$$R \stackrel{\triangle}{=} \sum_{i \not\succ j,\ j \not\succ i,\ k = \wp(i,j)} e_i \psi^*(k,i) \Upsilon(k) \psi(k,j) e_j^*$$

In the above e_i denotes a block-column vector with all zero entries except for the i^{th} block which is an identity matrix. R has non-zero (i,j) entry only when neither i nor j is the ancestor of the other. $\wp(i,j)$ denotes the nearest common ancestor for the i^{th} and j^{th} bodies, while $\wp(k)$ denotes the immediate parent of the k^{th} body.

The algorithm for the elements of Ω **in the paper does not include the elements of R**. The correct algorithm can be summarized as follows:

$$\Omega(i,j) = \begin{cases} \Upsilon(i) & \text{for } i = j \\ \psi^*(k,i)\Omega^*(j,k) & \text{for } i \prec k \preceq j, \quad k = \wp(i) \\ \Omega(i,k)\psi(k,j) & \text{for } i \succeq k \succ j, \quad k = \wp(j) \\ \psi^*(\wp(i),i)\Omega(\wp(i),\wp(j))\psi(\wp(j),j) & \text{for } i \not\succ j,\ j \not\succ i,\ k = \wp(i,j) \end{cases} \tag{A}$$

The paper is missing the last of the four entries above. The overall algorithm consists of computing the diagonal $\Upsilon(k)$ matrices in a base to tips recursion followed by recursions that start with the diagonal terms to compute the off-diagonal rows and columns terms.

A summary of the proof is as follows. First note that $\mathcal{E}_\psi$ and ψ can be expressed as

$$\mathcal{E}_\psi = \sum_k e_{\wp(k)} \psi(\wp(k),k) e_k^*, \qquad \psi = \sum_{l \succeq j} e_l \psi(l,j) e_j^*$$

Thus

$$\begin{aligned}
\operatorname{diag}\left\{\mathcal{E}_\psi^* \Upsilon \mathcal{E}_\psi\right\} &= \operatorname{diag}\left\{ \left\{\sum_j e_j \psi^*(\wp(j), j) e_{\wp(j)}^*\right\} \Upsilon \left\{\sum_k e_{\wp(k)} \psi(\wp(k), k) e_k^*\right\} \right\} \\
&= \operatorname{diag}\left\{ \sum_{j,k,\wp(j)=\wp(k)} e_j \psi^*(\wp(j), j) \Upsilon(\wp(j)) \psi(\wp(k), k) e_k^* \right\} \\
&= \sum_j e_j \psi^*(\wp(j), j) \Upsilon(\wp(j)) \psi(\wp(j), j) e_j^*
\end{aligned}$$

At the component level this is equivalent to

$$H^*(k) D^{-1}(k) H(k) = \Upsilon(k) - \psi^*(\wp(j), j) \Upsilon(\wp(j)) \psi(\wp(j), j)$$

from which Eq. (41) follows.

Now let us look at the elements of $\Omega = \psi^* H^* D^{-1} H \psi$.

$$\begin{aligned}
\Omega &= \left\{\sum_{m \succeq i} e_i \psi^*(m, i) e_m^*\right\} H^* D^{-1} H \left\{\sum_{l \succeq j} e_l \psi(l, j) e_j^*\right\} \\
&= \sum_{m \succeq i,j} e_i \psi^*(m, i) H^*(m) D^{-1}(m) H(m) \psi(m, j) e_j^*
\end{aligned}$$

Hence,

$$\Omega(i, j) = \sum_{m \succeq i,j} \psi^*(m, i) H^*(m) D^{-1}(m) H(m) \psi(m, j)$$

The expressions in Eq. (A) are a recursive form of the above expression for $\Omega(i, j)$.

References

[1] G. Rodriguez, A. Jain, and K. Kreutz-Delgado, "Spatial Operator Algebra for Multibody System Dynamics," *Journal of the Astronautical Sciences*, vol. 40, pp. 27–50, Jan.–March 1992.