Sensitivity Analysis of Multibody Dynamics Using Spatial Operators

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Abstract. This paper discusses an approach for sensitivity analysis of multibody dynamics using spatial operators. The spatial operators are rooted in the function space approach to estimation theory developed in the decades that followed the introduction of the Kalman filter. The operators provide a mathematical framework for studying a wide range of analytical and computational problems associated with multi-body system dynamics. This paper focuses on the computation of the sensitivity of the system mass matrix and develops an analytical expression for the same using spatial operators.

Key Words. Multibody dynamics, sensitivities, spatial operators, mass matrix, spatial algebra, robotics.

1 Introduction

Kalman introduced the notion of a state space, and a recursive filter [1] that computes the best estimate of the state from possibly noisy past measurements. The optimal Bryson [2] smoother computes the best state estimate using both past and future data. Although several authors seemed to have arrived at similar results at approximately the same time, Kailath [3, 4] was most likely the first to recognize many new techniques. He introduced the "innovations" approach, which when specialized to state space systems was a more advanced way to derive optimal linear estimators such as the Kalman filter. He also recognized the value to estimation theory of powerful mathematical techniques (Gohberg and Krein) to factor positive operators into a product of two closely related integral operators with triangular kernels. The function space approach reached maturity in the work of Balakrishnan [5], who introduced the elegant methods of Hilbert space. At the end of this period, we knew how to easily solve very complicated linear filtering problems using linear integral operators, operator factorization methods, and triangular (Volterra) factors. In the mid 1980's, the authors recognized [6–8] that the equations of mechanical systems had an almost perfect analogy to those of state space linear systems. Discovery of this analogy allowed the use in mechanics of very advanced methods and computational architectures (Kalman, Bryson, Riccati, etc.) that had emerged from estimation theory.

An overview of the spatial operator algebra can be found in [9–11]. Sensitivity computations are essential in problems involving optimization, linearization, nonlinear analysis etc. In practice, due to the complexity of the dynamics quantities, numerical differentiation techniques are often utilized for such sensitivity computations. Not only are these techniques non-robust, they also introduce errors and are computationally expensive. Analytical techniques to compute these sensitivities are typically limited to "small" problems. In this paper we describe our recent results which use an analytical approach for these sensitivity computations using spatial operators. The promise of this approach is that it is applicable to large-dimensional systems, is accurate and is computationally efficient. Example problems where such sensitivity computations are useful can be found in [12, 13]. For the purposes of this discussion we use a serial chain system with single degree of freedom hinges as our example problem.

1.1 Overview of Spatial Operators for Serial Chain Systems

The aim of this subsection is to summarize briefly the essential ideas underlying spatial operators leading up to the Newton-Euler Operator Factorization $\mathcal{M}(\theta) = \boldsymbol{H}\phi\boldsymbol{M}\phi^*\boldsymbol{H}$ of the manipulator mass matrix. While this is done here for a serial chain manipulator, the factorization results apply to a much more general class of complex joint-connected mechanical systems, including tree configurations with flexible links and joints [14].

Consider a serial manipulator with $\mathcal{N}$ rigid links. The links are numbered in increasing order from tip to base. The outer-most link is link 1, and the inner-most link is link $\mathcal{N}$. The overall number of degrees-of-freedom for the manipulator is $\mathcal{N}$. There are two joints attached to the k^{th} link. A coordinate frame $\mathcal{O}_k$ is attached to the inboard joint, and another frame $\mathcal{O}_{k-1}^+$ is attached to the outboard joint. Frame $\mathcal{O}_k$ is also the body frame for the k^{th} link. The k^{th} joint connects the $(k+1)^{st}$ and k^{th} links, and its motion is defined as the motion of frame $\mathcal{O}_k$ with respect to frame $\mathcal{O}_k^+$. When applicable, the free-space motion of a manipulator is modeled by attaching a 6 degree-of-freedom joint between the base link and the inertial frame about which the free-space motion occurs. However, in this paper, without loss of generality and for the sake of notational simplicity, all joints are assumed to be single rotational degree-of-freedom joints with the k^{th} joint coordinate given by $\theta(k)$. Extension to joints with more rotational and translational degrees-of-freedom is easy [15].

The transformation operator $\phi(k,k-1)$ between the $\mathcal{O}_{k-1}$ and $\mathcal{O}_k$ frames is

$$\phi(k,k-1) = \begin{pmatrix} \boldsymbol{I}_3 & \tilde{l}(k,k-1) \\ 0 & \boldsymbol{I}_3 \end{pmatrix} \in \mathcal{R}^{6\times 6}$$

where $l(k,k-1)$ is the vector from frame $\mathcal{O}_k$ to frame $\mathcal{O}_{(k-1)}$, and $\tilde{l}(k,k-1) \in \mathcal{R}^{3\times 3}$ is the skew–symmetric matrix associated with the cross-product operation.

The spatial velocity of the k^{th} body frame $\mathcal{O}_k$ is $V(k) = [\omega^*(k), v^*(k)]^* \in \mathcal{R}^6$, where $\omega(k)$ and $v(k)$ are the angular and linear velocities of $\mathcal{O}_k$. With $h(k) \in \mathcal{R}^3$ denoting the k^{th} joint axis vector, $\boldsymbol{H}(k) = [h^*(k), 0] \in \mathcal{R}^1 \times \mathcal{R}^6$ denotes the joint map matrix for the joint, and the relative spatial velocity across the k^{th} joint is $\boldsymbol{H}^*(k)\dot{\theta}(k)$. The spatial force of interaction $f(k)$ across the k^{th} joint is $f(k) = [N^*(k), F^*(k)]^* \in \mathcal{R}^6$, where $N(k)$ and $F(k)$ are the moment and force components respectively. The 6×6 spatial inertia matrix $\boldsymbol{M}(k)$ of the k^{th} link in the coordinate frame $\mathcal{O}_k$ is

$$\boldsymbol{M}(k) = \begin{pmatrix} \mathcal{J}(k) & m(k)\tilde{p}(k) \\ -m(k)\tilde{p}(k) & m(k)\boldsymbol{I}_3 \end{pmatrix}$$

where $m(k)$ is the mass, $p(k) \in \mathcal{R}^3$ is the vector from $\mathcal{O}_k$ to the k^{th} link center of mass, and $\mathcal{J}(k) \in \mathcal{R}^{3\times 3}$ is the rotational inertia of the k^{th} link about $\mathcal{O}_k$. $\boldsymbol{I}_3$ is the 3×3 unit matrix.

The recursive Newton–Euler equations are [6,16]

$$\left\{ \begin{array}{l} \quad V(\mathcal{N}+1) = 0; \qquad \alpha(\mathcal{N}+1) = 0 \\ \textbf{for } \boldsymbol{k} = \boldsymbol{\mathcal{N}} \cdots \mathbf{1} \\ \quad V(k) = \phi^*(k+1,k)V(k+1) + \boldsymbol{H}^*(k)\dot{\theta}(k) \\ \quad \alpha(k) = \phi^*(k+1,k)\alpha(k+1) + \boldsymbol{H}^*(k)\ddot{\theta}(k) \\ \qquad\qquad + a(k) \\ \textbf{end loop} \end{array} \right.$$

$$\left\{ \begin{array}{l} \quad f(0) = 0 \\ \textbf{for } \boldsymbol{k} = \mathbf{1} \cdots \boldsymbol{\mathcal{N}} \\ \quad f(k) = \phi(k,k-1)f(k-1) + \boldsymbol{M}(k)\alpha(k) \\ \qquad\qquad + b(k) \\ \quad \boldsymbol{T}(k) = \boldsymbol{H}(k)f(k) \\ \textbf{end loop} \end{array} \right.$$

where $\boldsymbol{T}(k)$ is the applied moment at joint k. The non-linear, velocity dependent terms $a(k)$ and $b(k)$ are respectively the Coriolis acceleration and the gyroscopic force terms for the k^{th} link.

The "stacked" notation $\theta = \text{col}\left\{\theta(k)\right\} \in \mathcal{R}^{\mathcal{N}}$ is used to simplify the above recursive Newton-Euler equations. This notation [9] eliminates the arguments k associated with the individual links by defining composite vectors, such as θ, which apply to the entire manipulator system. We define

$$\begin{array}{ll} \boldsymbol{T} = \text{col}\left\{\boldsymbol{T}(k)\right\} \in \mathcal{R}^{\mathcal{N}} & V = \text{col}\left\{V(k)\right\} \in \mathcal{R}^{6\mathcal{N}} \\ f = \text{col}\left\{f(k)\right\} \in \mathcal{R}^{6\mathcal{N}} & \alpha = \text{col}\left\{\alpha(k)\right\} \in \mathcal{R}^{6\mathcal{N}} \\ a = \text{col}\left\{a(k)\right\} \in \mathcal{R}^{6\mathcal{N}} & b = \text{col}\left\{b(k)\right\} \in \mathcal{R}^{6\mathcal{N}} \end{array}$$

In this notation, the equations of motion are [6,7]:

$$V = \phi^*\boldsymbol{H}^*\dot{\theta}; \qquad \alpha = \phi^*[\boldsymbol{H}^*\ddot{\theta} + a] \tag{1.1}$$

$$f = \phi[\boldsymbol{M}\alpha + b]; \qquad \boldsymbol{T} = \boldsymbol{H}f = \mathcal{M}\ddot{\theta} + \mathcal{C} \tag{1.2}$$

where the mass matrix $\mathcal{M}(\theta) = \boldsymbol{H}\phi\boldsymbol{M}\phi\boldsymbol{H}^*$; $\mathcal{C}(\theta,\dot{\theta}) = \boldsymbol{H}\phi[\boldsymbol{M}\phi^* a + b] \in \mathcal{R}^{\mathcal{N}}$ is the Coriolis term; $\boldsymbol{H} = \text{diag}\left\{\boldsymbol{H}(k)\right\} \in \mathcal{R}^{\mathcal{N}\times 6\mathcal{N}}$; $\boldsymbol{M} = \text{diag}\left\{\boldsymbol{M}(k)\right\} \in \mathcal{R}^{6\mathcal{N}\times 6\mathcal{N}}$; and $\phi \in \mathcal{R}^{6\mathcal{N}\times 6\mathcal{N}}$

$$\phi = (\boldsymbol{I} - \mathcal{E}_\phi)^{-1} = \begin{pmatrix} \boldsymbol{I} & 0 & \ldots & 0 \\ \phi(2,1) & \boldsymbol{I} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \phi(n,1) & \phi(n,2) & \ldots & \boldsymbol{I} \end{pmatrix} \tag{1.3}$$

with $\phi(i,j) = \phi(i,i-1)\cdots\phi(j+1,j)$ for $i > j$. The shift operator $\mathcal{E}_\phi \in \mathcal{R}^{6\mathcal{N}\times 6\mathcal{N}}$ is defined as

$$\mathcal{E}_\phi = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \phi(2,1) & 0 & \ldots & 0 & 0 \\ 0 & \phi(3,2) & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \phi(\mathcal{N},\mathcal{N}-1) & 0 \end{pmatrix} \tag{1.4}$$

Using spatial operators one can obtain operator factorizations of the mass matrix and its inverse as follows:

Identity 1.1

$$\begin{aligned}\mathcal{M} &= \boldsymbol{H}\phi\boldsymbol{M}\phi^*\boldsymbol{H}^* \\ \mathcal{M} &= [\boldsymbol{I}+\boldsymbol{H}\phi\boldsymbol{K}]\boldsymbol{D}[\boldsymbol{I}+\boldsymbol{H}\phi\boldsymbol{K}]^* \\ [\boldsymbol{I}+\boldsymbol{H}\phi\boldsymbol{K}]^{-1} &= \boldsymbol{I}-\boldsymbol{H}\psi\boldsymbol{K} \\ \mathcal{M}^{-1} &= [\boldsymbol{I}-\boldsymbol{H}\psi\boldsymbol{K}]^*\boldsymbol{D}^{-1}[\boldsymbol{I}-\boldsymbol{H}\psi\boldsymbol{K}]\end{aligned}$$

However we will not go any further in this direction, but instead will focus on the process of computing the sensitivity of spatial operators. For the purposes of this discussion we have focused our attention on serial chains with single degree of freedom. We will maintain this focus in the rest of this paper though the generalization to general tree topology systems and hinges is straightforward.

2 Sensitivity Computations

Given the generalized coordinates vector θ and a multi-valued function $g(\theta)$, our general approach to computing its sensitivity will be to first compute an expression for its time derivative $\dot{g}(\theta)$ and then use the relationship

$$\dot{g}(\theta) = \frac{\partial g(\theta)}{\partial \theta}\dot{\theta}$$

to obtain $\frac{\partial g(\theta)}{\partial \theta_i}$ from the i^{th} column of $\frac{\partial g(\theta)}{\partial \theta}$.

2.1 The Shift operator $\mathbb{S}$

We first introduce and define the *shift operator*, $\mathbb{S} \in \mathcal{R}^{6\mathcal{N}\times 6\mathcal{N}}$ consisting of $\mathcal{R}^{6\times 6}$ block elements with the only non-zero ones being the identity $\mathcal{R}^{6\times 6}$ elements along the first sub-diagonal. Also we define the δ_{cond} notation such that its value is 1 if *cond* is true and is 0 otherwise.

Some useful properties of the shift operator $\mathbb{S}$ are defined in the following lemma.

Lemma 2.1 **: Properties of the shift operator $\mathbb{S}$**

Given block diagonal matrices A and B, the following relationships hold:

$$\begin{aligned}(\mathbb{S}A\mathbb{S}^*)\mathbb{S}B &= \mathbb{S}AB \\ (\mathbb{S}^*A\mathbb{S})\mathbb{S}^*B &= \mathbb{S}^*AB \\ A\mathbb{S}^*(\mathbb{S}B\mathbb{S}^*) &= AB\mathbb{S}^* \\ A\mathbb{S}(\mathbb{S}^*B\mathbb{S}) &= AB\mathbb{S} \\ (\mathbb{S}A\mathbb{S}^*)(\mathbb{S}B\mathbb{S}^*) &= \mathbb{S}AB\mathbb{S}^* \\ (\mathbb{S}^*A\mathbb{S})(\mathbb{S}^*B\mathbb{S}) &= \mathbb{S}^*AB\mathbb{S}\end{aligned}$$

Special Cases:

$$\begin{aligned}(\mathbb{S}\mathbb{S}^*)\mathbb{S}A &= \mathbb{S}A & \mathbb{S}A(\mathbb{S}^*\mathbb{S}) &= \mathbb{S}A \\ (\mathbb{S}^*\mathbb{S})\mathbb{S}^*A &= \mathbb{S}^*A & \mathbb{S}^*A(\mathbb{S}\mathbb{S}^*) &= \mathbb{S}^*A \\ (\mathbb{S}^*\mathbb{S})A\mathbb{S}^* &= A\mathbb{S}^* & A(\mathbb{S}^*\mathbb{S})\mathbb{S}^* &= A\mathbb{S}^* \\ (\mathbb{S}\mathbb{S}^*)A\mathbb{S} &= A\mathbb{S} & A(\mathbb{S}\mathbb{S}^*)\mathbb{S} &= A\mathbb{S}\end{aligned}$$

Proof: *Use direct evaluation to verify these identities.* ∎

2.2 The $\mathbb{H}^i$, $\mathbb{H}^i_s$, and $\mathbb{H}^i_\delta$ operators

We define $\mathbb{H}(i)$ as

$$\mathbb{H}(i) \;\triangleq\; \mathcal{S}^*\left[\boldsymbol{H}^*(k)\right] = \begin{pmatrix}\tilde{h}(i) & 0 \\ 0 & \tilde{h}(i)\end{pmatrix} \tag{2.5}$$

with

$$\boldsymbol{H}^*(i) = \begin{bmatrix}h(i) \\ 0\end{bmatrix} \tag{2.6}$$

$\mathbb{H}^i_s$ is the block diagonal matrix defined as $\mathbb{H}^i_s(k,k) = \mathbb{H}(i)\delta_{k<i}$

$$\mathbb{H}^i_s(k,k) = \begin{cases}\mathbb{H}(i) & \text{for } k>i \\ 0 & \text{for } k\le i\end{cases} \tag{2.7}$$

We similarly also define the block diagonal matrices $\mathbb{H}^i$ and $\mathbb{H}^i_\delta$ as having $\mathbb{H}(i)$ along the block diagonal in the following manner:

$$\mathbb{H}^i(k,k) = \mathbb{H}(i)\delta_{k\le i}, \quad \text{and} \quad \mathbb{H}^i_\delta(k,k) = \mathbb{H}(i)\delta_{k=i} \tag{2.8}$$

There is an important new quantity in this result, and it has a simple physical interpretation. The matrix $\mathbb{H}^i_\delta$ is the $6\mathcal{N}\times 6\mathcal{N}$ matrix whose elements are all zero, except for a single 6×6 block $\mathbb{H}(i)$ at the i^{th} location on the diagonal. The index i corresponds to the joint-angle θ_i with respect to which the sensitivity $\mathcal{M}_{\theta_i}$ is being taken.

Note that

$$\mathbb{H}^i = \mathbb{H}^i_s + \mathbb{H}^i_\delta, \quad \text{and} \quad \mathbb{H}^i_s = \mathbb{S}^*\mathbb{H}^i\mathbb{S} \tag{2.9}$$

Also, $\mathbb{H}^i$, $\mathbb{H}^i_s$ and $\mathbb{H}^i_\delta$ are all skew-symmetric.

Lemma 2.2 **: Composition of $\mathbb{H}^i_\delta$ etc. with arbitrary matrices.**

Show that for a given matrix X we have that,

$$\begin{aligned}[\mathbb{H}^i_s X](k,j) &= \mathbb{H}(i)X(k,j)\delta_{k<i} \\ [X\mathbb{H}^i_s](k,j) &= X(k,j)\mathbb{H}(i)\delta_{j<i} \\ [\mathbb{H}^i X](k,j) &= \mathbb{H}(i)X(k,j)\delta_{k\le i} \\ [X\mathbb{H}^i](k,j) &= X(k,j)\mathbb{H}(i)\delta_{j\le i} \\ [\mathbb{H}^i_\delta X](k,j) &= \mathbb{H}(i)X(k,j)\delta_{k=i} \\ [X\mathbb{H}^i_\delta](k,j) &= X(k,j)\mathbb{H}(i)\delta_{j=i} \\ [X\mathbb{H}^i_\delta Y](k,j) &= X(k,i)\mathbb{H}(i)Y(i,j)\end{aligned} \tag{2.10}$$

Proof: *These identities are established by simply evaluating the products on the right hand side of the equations.* ∎

Define

$$\tilde{\Omega}(k) \triangleq \begin{pmatrix} \tilde{\omega}(k) & 0 \\ 0 & \tilde{\omega}(k) \end{pmatrix} \tag{2.11}$$

and

$$\tilde{\Omega} = \sum_{i=1}^{n} \mathbb{H}^i \dot{\theta}(i), \quad \tilde{\Omega}_s = \sum_{i=1}^{n} \mathbb{H}_s^i \dot{\theta}(i), \quad \tilde{\Omega}_\delta = \sum_{i=1}^{n} \mathbb{H}_\delta^i \dot{\theta}(i)$$

$\tilde{\Omega}$ is the spatial cross product matrix associated with the spatial vector $\Omega(k)$, where $\Omega(k)$ is defined as:

$$\Omega(k) \triangleq \begin{bmatrix} \omega(k) \\ 0 \end{bmatrix} \tag{2.12}$$

Define also the quantities $\tilde{\Omega}(k) \in \mathcal{R}^6$ and $\tilde{\Omega} \in \mathcal{R}^{6\mathcal{N}}$ as follows:

$$\begin{aligned} \tilde{\Omega}(k) &\triangleq \Omega(k) - \Omega(k+1) = \boldsymbol{H}^*(k)\dot{\theta}(k) \\ \tilde{\Omega} &\triangleq \operatorname{col}\left\{\tilde{\Omega}(k)\right\} \end{aligned} \tag{2.13}$$

Note that

$$\tilde{\Omega} = \tilde{\Omega}_s + \tilde{\Omega}_\delta, \quad \text{and} \quad \tilde{\Omega}_s = \mathbb{S}^* \tilde{\Omega} \mathbb{S} \tag{2.14}$$

2.3 Sensitivities of $\phi(k+1,k)$, $\boldsymbol{H}(k)$ and $\boldsymbol{M}(k)$

***Lemma 2.3* : Time derivatives of $\phi(k+1,k)$, $\boldsymbol{H}(k)$ and $\boldsymbol{M}(k)$**
We have that

$$\dot{\boldsymbol{M}}(k) = \tilde{\Omega}(k)\boldsymbol{M}(k) - \boldsymbol{M}(k)\tilde{\Omega}(k) \tag{2.15}$$

$$\dot{\boldsymbol{H}}^*(k) = \tilde{\Omega}(k+1)\boldsymbol{H}^*(k) \tag{2.16}$$

$$\begin{aligned} \dot{\phi}(k+1,k) = {} & \tilde{\Omega}(k+1)\phi(k+1,k) \\ & - \phi(k+1,k)\tilde{\Omega}(k+1) \end{aligned} \tag{2.17}$$

Proof:

$$\dot{\phi}(k+1,k) = \begin{pmatrix} 0 & \tilde{\omega}(k+1)\widetilde{\ell(k+1,k)} \\ 0 & 0 \end{pmatrix} \tag{2.18}$$

$$= \begin{pmatrix} 0 & \tilde{\omega}(k+1)\tilde{\ell}(k+1,k) - \tilde{\ell}(k+1,k)\tilde{\omega}(k+1) \\ 0 & 0 \end{pmatrix} \tag{2.19}$$

Also,

$$\dot{\boldsymbol{H}}^*(k) = \begin{bmatrix} \tilde{\omega}(k+1)h(k) \\ 0 \end{bmatrix} = \tilde{\Omega}(k+1)\boldsymbol{H}^*(k) \tag{2.20}$$

Also,

$$\begin{aligned} \dot{\boldsymbol{M}}(k) &= \begin{pmatrix} \tilde{\omega}(k)\mathcal{J}(k) - \mathcal{J}(k)\tilde{\omega}(k) & m(k)[\tilde{\omega}(k)p(k)]^{\sim} \\ -m(k)[\tilde{\omega}(k)p(k)]^{\sim} & 0 \end{pmatrix} \\ &= \tilde{\Omega}(k)\boldsymbol{M}(k) - \boldsymbol{M}(k)\tilde{\Omega}(k) \end{aligned} \tag{2.21}$$

∎

***Lemma 2.4* : Sensitivities of $\phi(k+1,k)$, $\boldsymbol{H}(k)$ and $\boldsymbol{M}(k)$**

$$\begin{aligned} [\phi(k+1,k)]_{\theta_i} &= [\mathbb{H}(i)\phi(k+1,k) - \phi(k+1,k)\mathbb{H}(i)] \cdot \delta_{k<i} \\ &= \begin{cases} 0 & \textit{for} \quad k \geq i \\ \mathbb{H}(i)\phi(k+1,k) - \phi(k+1,k)\mathbb{H}(i) & \textit{for} \quad k > i \end{cases} \end{aligned} \tag{2.22}$$

$$\begin{aligned} [\boldsymbol{H}^*(k)]_{\theta_i} &= \mathbb{H}(i)\boldsymbol{H}^*(k)\delta_{k<i} \\ &= \begin{cases} 0 & \textit{for} \quad k \geq i \\ \begin{bmatrix} \tilde{h}(i)h(k) \\ 0 \end{bmatrix} & \textit{for} \quad k < i \end{cases} \end{aligned} \tag{2.23}$$

$$\begin{aligned} [\boldsymbol{M}(k)]_{\theta_i} &= [\mathbb{H}(i)\boldsymbol{M}(k) - \boldsymbol{M}(k)\mathbb{H}(i)]\delta_{k\leq i} \\ &= \begin{cases} 0 & \textit{for} \quad k > i \\ \mathbb{H}(i)\boldsymbol{M}(k) - \boldsymbol{M}(k)\mathbb{H}(i) & \textit{for} \quad k \leq i \end{cases} \end{aligned} \tag{2.24}$$

Proof: *Follow directly from Lemma 2.3.* ∎

2.4 Operator sensitivities of ϕ, $\boldsymbol{H}$, M

Define the operator Δ_ϕ as follows.

$$\Delta_\phi = \begin{pmatrix} \phi(2,1) & 0 & \dots & \\ 0 & \phi(3,2) & & \\ \vdots & & \ddots & \\ 0 & \dots & \dots & \phi(n+1,n) \end{pmatrix} \tag{2.25}$$

Note that

$$\mathcal{E}_\phi = \mathbb{S}\Delta_\phi \tag{2.26}$$

***Lemma 2.5* : Time Derivatives of Spatial Operators**

$$\dot{\Delta}_\phi = \tilde{\Omega}_s \Delta_\phi - \Delta_\phi \tilde{\Omega}_s \tag{2.27}$$

$$\dot{\mathcal{E}}_\phi = \tilde{\Omega}\mathcal{E}_\phi - \mathcal{E}_\phi \tilde{\Omega}_s \tag{2.28}$$

$$\dot{\boldsymbol{H}}^* = \tilde{\Omega}_s \boldsymbol{H}^* \tag{2.29}$$

$$\dot{\boldsymbol{M}} = \tilde{\Omega}\boldsymbol{M} - \boldsymbol{M}\tilde{\Omega} \tag{2.30}$$

$$\begin{aligned} \dot{\phi} &= \phi\tilde{\Omega}\tilde{\phi} - \tilde{\phi}\tilde{\Omega}_s\phi \\ &= \phi\tilde{\Omega}_\delta\phi + \tilde{\Omega}_s\phi - \phi\tilde{\Omega} \end{aligned} \tag{2.31}$$

Proof: *Eq. 2.27 can be derived by assembling the component time derivatives of Eq. 2.25 from Eq. 2.17. Eq. 2.29 follows by applying the identities in Lemma 2.1 to Eq. 2.27. Eq. 2.29 and Eq. 2.30 are simply matrix versions of Eq. 2.16 and Eq. 2.15 respectively. For Eq. 2.31 we have that*

$$\begin{aligned}\dot{\phi} &= -\phi\dot{\overline{\phi^{-1}}}\phi = -\phi[I - \dot{\overline{\mathcal{E}_\phi}}]\phi \\ &= \phi\dot{\mathcal{E}}_\phi\phi = \phi[\tilde{\mathbb{\Omega}}\mathcal{E}_\phi - \mathcal{E}_\phi\tilde{\mathbb{\Omega}}_s]\phi \\ &= \phi\tilde{\mathbb{\Omega}}\tilde{\phi} - \tilde{\phi}\tilde{\mathbb{\Omega}}_s\phi\end{aligned}$$

■

Lemma 2.6 **: Operator sensitivities of ϕ, $\boldsymbol{H}$, M**

$$\begin{aligned}[\Delta_\phi]_{\theta_i} &= \mathbb{H}_s^i\Delta_\phi - \Delta_\phi\mathbb{H}_s^i && (2.32)\\ [\mathcal{E}_\phi]_{\theta_i} &= \mathbb{H}^i\mathcal{E}_\phi - \mathcal{E}_\phi\mathbb{H}_s^i && (2.33)\\ [\phi]_{\theta_i} &= \phi\mathbb{H}_\delta^i\phi - \phi\mathbb{H}^i + \mathbb{H}_s^i\phi && (2.34)\\ [\phi]_{\theta_i}(k,j) &= [\phi(k,i)\mathbb{H}(i)\phi(i,j)\delta_{k\geq i} - \phi(k,j)\mathbb{H}(i) \\ &\quad + \mathbb{H}(i)\phi(k,j)\delta_{k<i}]\delta_{j<i} && (2.35)\\ [\boldsymbol{H}^*]_{\theta_i} &= \mathbb{H}_s^i\boldsymbol{H}^* && (2.36)\\ [\boldsymbol{M}]_{\theta_i} &= \mathbb{H}^i M - \boldsymbol{M}\mathbb{H}^i && (2.37)\end{aligned}$$

Proof:

$$[\mathcal{E}_\phi]_{\theta_i} = \mathbb{S}[\Delta_\phi]_{\theta_i}$$

Also, since $\phi = [I - \mathcal{E}_\phi]^{-1}$

$$\begin{aligned}[\phi]_{\theta_i} &= -\phi[\phi^{-1}]_{\theta_i}\phi = \phi[\mathbb{H}^i\mathcal{E}_\phi - \mathcal{E}_\phi\mathbb{H}_s^i]\phi \\ &= \phi\mathbb{H}^i\tilde{\phi} - \tilde{\phi}\mathbb{H}_s^i\phi = \phi\mathbb{H}_\delta^i\phi - \phi\mathbb{H}^i + \mathbb{H}_s^i\phi\end{aligned}$$

$$\begin{aligned}[\phi]_{\theta_i}(k,j) &= \phi(k,i)\mathbb{H}(i)\phi(i,j)\delta_{k\geq i\geq j} \\ &\quad - \phi(k,j)\mathbb{H}(i)\delta_{j\leq i} + \mathbb{H}(i)\phi(k,j)\delta_{k<i} \\ &= [\phi(k,i)\mathbb{H}(i)\phi(i,j)\delta_{k\geq i} \\ &\quad - \phi(k,j)\mathbb{H}(i) + \mathbb{H}(i)\phi(k,j)\delta_{k<i}]\delta_{j<i}\end{aligned}$$

■

Lemma 2.7 **: Sensitivity of $H\phi$**

$$\begin{aligned}[\dot{\boldsymbol{H}\phi}] &= \boldsymbol{H}\phi[\tilde{\mathbb{\Omega}}_\delta\phi - \tilde{\mathbb{\Omega}}] \\ [\boldsymbol{H}\phi]_{\theta_i} &= \boldsymbol{H}\phi[\mathbb{H}_\delta^i\phi - \mathbb{H}^i]\end{aligned} \tag{2.38}$$

Proof:

$$[\boldsymbol{H}\phi]_{\theta_i} \overset{2.34,2.36}{=} \boldsymbol{H}_{\theta_i}\phi + \boldsymbol{H}\phi_{\theta_i} = \boldsymbol{H}\phi[\mathbb{H}_\delta^i\phi - \mathbb{H}^i]$$

■

3 Mass Matrix Related Quantities

3.1 Sensitivity of $\phi\boldsymbol{M}\phi^*$

Lemma 3.1 **: Sensitivity of $\phi\boldsymbol{M}\phi^*$**

$$[\phi\boldsymbol{M}\phi^*]_{\theta_i} = [\phi\mathbb{H}_\delta^i + \mathbb{H}_s^i]\phi\boldsymbol{M}\phi^* - \phi\boldsymbol{M}\phi^*[\mathbb{H}_\delta^i\phi^* + \mathbb{H}_s^i] \tag{3.39}$$

Proof:

$$\begin{aligned}[\phi\boldsymbol{M}\phi^*]_{\theta_i} &= [\phi]_{\theta_i}\boldsymbol{M}\phi^* + \phi M[\phi]_{\theta_i}^* + \phi\boldsymbol{M}_{\theta_i}\phi^* \\ &= [\phi\mathbb{H}^i\tilde{\phi} - \tilde{\phi}\mathbb{H}_s^i\phi]\boldsymbol{M}\phi^* \\ &\quad + \phi M[\phi^*\mathbb{H}_s^i\tilde{\phi}^* - \tilde{\phi}^*\mathbb{H}^i\phi^*] \\ &\quad + \phi[\mathbb{H}^i\boldsymbol{M} - \boldsymbol{M}\mathbb{H}^i]\phi^* \\ &= [\phi\mathbb{H}_\delta^i + \mathbb{H}_s^i]\phi\boldsymbol{M}\phi^* - \phi\boldsymbol{M}\phi^*[\mathbb{H}_\delta^i\phi^* + \mathbb{H}_s^i]\end{aligned}$$

■

3.2 Sensitivity of the Mass Matrix $\mathcal{M}_{\theta_i}$

Lemma 3.2 **: Sensitivity of the Mass Matrix** $\mathcal{M}_i = [\boldsymbol{H}\phi\boldsymbol{M}\phi^*\boldsymbol{H}^*]_{\theta_i}$

$$\mathcal{M}_{\theta_i} = \boldsymbol{H}\phi[\mathbb{H}_\delta^i\phi\boldsymbol{M} - \boldsymbol{M}\phi^*\mathbb{H}_\delta^i]\phi^*\boldsymbol{H}^* \tag{3.40}$$

Proof:

$$\begin{aligned}\mathcal{M}_{\theta_i} &= \boldsymbol{H}_{\theta_i}\phi\boldsymbol{M}\phi^*\boldsymbol{H}^* + \boldsymbol{H}\phi\boldsymbol{M}\phi^*\boldsymbol{H}_{\theta_i}^* + H[\phi\boldsymbol{M}\phi^*]_{\theta_i}\boldsymbol{H}^* \\ &= \boldsymbol{H}\phi[\mathbb{H}_\delta^i\phi\boldsymbol{M} - \boldsymbol{M}\phi^*\mathbb{H}_\delta^i]\phi^*\boldsymbol{H}^*\end{aligned}$$

■

The above formula in Eq. 3.40 is closed-form, in the sense that it explicitly computes the mass matrix sensitivity in terms of the operators ϕ, $\boldsymbol{M}$, and $\boldsymbol{H}$ appearing in the mass matrix itself. That the formula is closed-form is of extreme importance, because it implies that the mass matrix derivatives can be easily computed using operations and spatially recursive algorithms similar to those used to compute the mass matrix itself.

We state below without proof an alternative expression for the sensitivity of the mass matrix using articulated body inertia quantities.

Lemma 3.3 **: Alternative expression for $\mathcal{M}_{\theta_i}$**
Note that since $\phi\boldsymbol{M}\phi^*\boldsymbol{H}^* = [I + \phi\boldsymbol{K}\boldsymbol{H}]\boldsymbol{P}\phi^*\boldsymbol{H}^*$,

$$\begin{aligned}\mathcal{M}_{\theta_i} &= \boldsymbol{H}\phi[\mathbb{H}_\delta^i(I + \phi\boldsymbol{K}\boldsymbol{H})\boldsymbol{P} \\ &\quad - \boldsymbol{P}(I + \phi\boldsymbol{K}\boldsymbol{H})^*\mathbb{H}_\delta^i]\phi^*\boldsymbol{H}^*\end{aligned}$$

■

4 Concluding Remarks

This overview aimed at presenting the use of spatial operators for studying sensitivities of multibody dynamics quantities. The general approach is made possible by the very high-level of mathematical abstraction allowed by spatial operators.

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