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An Analysis of the Kinematics and Dynamics of Under–Actuated Manipulators

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Abstract

In this paper we study the dynamics and kinematics of manipulators that have fewer actuators than degrees of freedom. These under–actuated manipulators arise in a number of important applications such as free–flying space robots, hyper–redundant manipulators, manipulators with structural flexibility, etc. In our analysis we decompose such under–actuated manipulators into component active and passive arms. This decomposition allows techniques previously developed for regular (fully–actuated) manipulators to be applied to under–actuated systems. Spatial operator identities are used to develop closed–form expressions for the generalized accelerations for the system. These expressions form the basis for a recursive $O(\mathcal{N})$ dynamics algorithm. The structure of this algorithm is a hybrid of known forward and inverse dynamics algorithms for regular manipulators. We also develop expressions and computational algorithms for the generalized and disturbance Jacobians for under–actuated manipulators. The application of the results in this paper to space manipulators is also described.

1 Introduction

An extensive amount of research on the kinematics, dynamics and control of robots has been carried out for *regular* (i.e. *fully–actuated*) manipulators. Every degree of freedom is an *active* degree of freedom for these manipulators. That is, for each degree of freedom, there is an independent generalized force that can be applied by a control actuator. However, many important applications involve manipulators with *passive* degrees of freedom, i.e., degrees of freedom with no corresponding control actuators. A passive degree of freedom can arise from either the absence or failure of an actuator, or due to a mode of operation that avoids the use of some available actuators. We refer to manipulators with passive degrees of freedom as *under–actuated* manipulators. For under–actuated manipulators, the number of available control actuators (or more specifically – the number of independent generalized forces) is less than the number of degrees of freedom.

The analysis of the dynamics of under–actuated manipulators is significantly more complex than that for regular manipulators. There is inertial coupling between the motion of the active and the passive hinges, so that mappings such as the Jacobian matrix, depend not only on the kinematical properties, but also on the inertia properties of the links. The presence of passive degrees of freedom often results in a lack of full controllability of the system. Previous work on the modeling and control of such manipulators can be found in references [1–3]. Some examples of under-actuated manipulators are described below.

1. Free–flying space manipulators possess six degrees of freedom for the base–body in addition to the manipulator hinge degrees of freedom. The six base–body degrees of freedom are controlled by an attitude and translation control system while the manipulator motion is controlled by actuators at the hinges. The manipulator is sometimes operated with the base–body control system turned off to conserve fuel. In this mode of operation, the six base–body degrees of freedom are passive while the manipulator hinge degrees of freedom are active degrees of freedom.

2. The improved dexterity and maneuverability provided by additional degrees of freedom has motivated the study of hyper–redundant and snake–like robots [1, 4]. It has been proposed that the mass of hyper–redundant manipulators be reduced by providing actuators at only some of the hinges while keeping the remaining hinges passive.

3. Flexible–link manipulators are inherently under–actuated. In addition to the hinge degrees of freedom, these manipulators possess deformation degrees of freedom from link flexibility. While careful structural analysis can provide good models for the elastic forces, these generalized forces cannot be directly controlled. As a result the deformation degrees of freedom represent passive degrees of freedom.

4. Actuator failure can convert an active hinge into a passive one. In the face of actuator failures, some degree of fault–tolerant control is highly desirable for robots in remote or hazardous environments. This requires the control of an under–actuated manipulator.

5. During multi–arm manipulation of task objects, the degrees of freedom associated with loose grasp contacts (eg. rolling contacts), or internal degrees of freedom of task objects (eg., shears, plungers) are typically passive degrees of freedom.

6. Fuel slosh has a significant impact on the dynamics of space vehicles. The complex models for fuel slosh are typically approximated to first order by pendulum models. These pendulum degrees of freedom represent passive degrees of freedom.

Research in these areas has resulted in the development of useful, though largely application–specific techniques for the analysis and control of these systems. The extensibility of these techniques to other types of under–actuated manipulators is not always obvious. For instance, most analysis of free–flying space–robots relies extensively on the non–holonomic constraint arising from the conservation of linear and angular momenta for these manipulators. These techniques cannot be applied to under–actuated systems such as hyper–redundant manipulators or flexible link manipulators for whom such momentum constraints do not hold. A goal of this paper is to take steps towards a more general framework for the kinematics and dynamics of under–actuated manipulators.

We make extensive use of techniques from the *spatial operator algebra* [5]. In Section 3, we review the spatial operator approach and develop the equations of motion for regular manipulators The modeling and dynamics of under–actuated manipulators is described in Section 4 Operator expressions for the generalized accelerations form the basis for a recursive $O(\mathcal{N})$ dynamics algorithm described in Section 5 Expressions and computational algorithms for the disturbance and generalized Jacobians that relate the motion of the active hinges to the motion of the passive hinges and the end–effector are developed in Section 6 The application of the results of this paper to space manipulators is discussed in Section 7

2 Nomenclature

Coordinate free spatial notation is used throughout this paper (see references [5, 6] for additional details). The notation $\tilde{l}$ denotes the cross–product matrix associated with the 3–dimensional vector l, while x^* denotes the transpose of a matrix x. In the stacked notation used in this paper, indices are used to identify quantities pertinent to a specific link. Thus for instance, V denotes the vector of the spatial velocities for all the links, and $V(k)$ denotes the spatial velocity vector for the k^{th} link. Some key quantities used in this paper are defined below.

n number of links in the manipulator

$\mathcal{O}_k$ inboard (body) frame for the k^{th} link

$\mathcal{O}_k^+$ outboard frame on the $(k+1)^{th}$ link

$r(k)$ number of degrees of freedom for the k^{th} hinge

$\mathcal{N}$ $= \sum_{k=1}^{n} r(k)$, the total number of degrees of freedom for the manipulator

$\theta(k)$ $\in \Re^{r(k)}$, the vector of generalized coordinates for the k^{th} hinge

$\beta(k)$ $\in \Re^{r(k)}$, the vector of generalized velocities for the k^{th} hinge

$l(k,j)$ $\in \Re^3$, the vector from the k^{th} to the j^{th} body frame

$\phi(k,j)$ $\triangleq \begin{pmatrix} I & \tilde{l}(k,j) \\ 0 & I \end{pmatrix} \in \Re^{6\times 6}$, the spatial transformation operator between the j^{th} and the k^{th} hinges

$H^*(k)$ $\in \Re^{6\times r(k)}$, the joint map matrix for the k^{th} hinge

$m(k)$ the mass of the k^{th} link

$p(k)$ $\in \Re^3$, the vector from $\mathcal{O}_k$ to the center of mass of the k^{th} link

$\mathcal{J}(k)$ $\in \Re^{3\times 3}$, the inertia matrix for the k^{th} link referred to $\mathcal{O}_k$

$M(k)$ $= \begin{pmatrix} \mathcal{J}(k) & m(k)\tilde{p}(k) \\ -m(k)\tilde{p}(k) & m(k)I_3 \end{pmatrix} \in \Re^{6\times 6}$, the spatial inertia of the k^{th} link referred to $\mathcal{O}_k$

$V(k)$ $= \begin{pmatrix} \omega(k) \\ v(k) \end{pmatrix} \in \Re^6$, the spatial velocity of the k^{th} link referred to $\mathcal{O}_k$, with $\omega(k)$ and $v(k)$ denoting the angular and linear velocity components respectively

$a(k)$ $= \begin{pmatrix} \tilde{\omega}(k+1) & 0 \\ 0 & \tilde{\omega}(k+1) \end{pmatrix} [V(k) - V(k+1)] \in \Re^6$, the Coriolis acceleration for the k^{th} link referred to $\mathcal{O}_k$

$b(k)$ $= \begin{pmatrix} \tilde{\omega}(k)\mathcal{J}(k)\omega(k) \\ m(k)\tilde{\omega}(k)\tilde{\omega}(k)p(k) \end{pmatrix} \in \Re^6$, the gyroscopic force for the k^{th} link referred to $\mathcal{O}_k$

$\alpha(k)$ $\in \Re^6$, the spatial acceleration of the k^{th} link referred to $\mathcal{O}_k$

$f(k)$ $= \begin{pmatrix} N(k) \\ F(k) \end{pmatrix} \in \Re^6$, the spatial force of interaction between the $(k+1)^{th}$ and the k^{th} link referred to $\mathcal{O}_k$, with $N(k)$ and $F(k)$ denoting the moment and force components respectively

$T(k)$ $\in \Re^{r(k)}$, the generalized force for the k^{th} hinge

$\mathcal{M}$ $\in \Re^{\mathcal{N}\times\mathcal{N}}$, the mass matrix for the manipulator

$\mathcal{C}$ $\in \Re^{\mathcal{N}}$, the vector of Coriolis and gyroscopic forces for the manipulator

$\mathcal{A}_p$, $\mathcal{A}_a$ the passive and active manipulator subsystems of an under–actuated manipulator

n_p, n_a the number of passive and active hinges

$\mathcal{N}_p$, $\mathcal{N}_a$ the number of passive and active degrees of freedom

$\mathcal{I}_p$, $\mathcal{I}_a$ the set of indices of the passive and active hinges

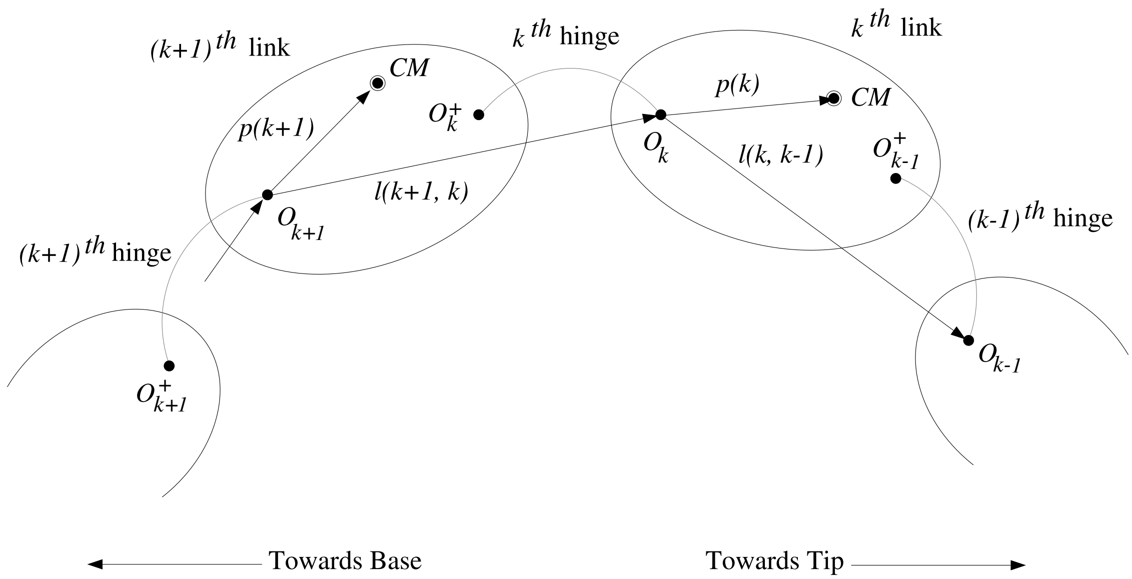

Figure 1: Illustration of the links and hinges in a serial manipulator

3 Dynamics of Regular Manipulators

We consider a serial manipulator with n rigid body links. As shown in Figure 3, the links are numbered in increasing order from tip to base. The outer most link is denoted link 1 and the inner most link is denoted link n.

Corresponding to each of the pair of (inboard and outboard) hinges attached to the k^{th} link, we assign two frames denoted $\mathcal{O}_k$ and $\mathcal{O}_{k-1}^+$ to the link. Frame $\mathcal{O}_k$ is on the inboard side and is also the body frame for the k^{th} link. The k^{th} hinge connects the $(k+1)^{th}$ and k^{th} links and its motion is defined as the motion of frame $\mathcal{O}_k$ with respect to frame $\mathcal{O}_{k+1}^+$. When applicable, the free space motion of a manipulator is modeled by attaching a 6 degree of freedom hinge between the base link and the inertial frame. The k^{th} hinge is assumed to have $r(k)$ degrees of freedom where $1 \le r(k) \le 6$, and its vector of generalized coordinates is denoted $\theta(k)$. For simplicity, and without any loss in generality, we assume that the number of generalized velocities for the hinge is also $r(k)$, i.e., there are no nonholonomic constraints on the hinge. The vector of generalized velocities for the k^{th} hinge is denoted $\beta(k) \in \Re^{r(k)}$. The choice of the hinge angle rates $\dot{\theta}(k)$ for the generalized velocities $\beta(k)$ is often an obvious and convenient choice. However, when the number of hinge degrees of freedom is greater than one, alternative choices are often preferred since they simplify and decouple the kinematic and dynamic parts of the equations of motion. An instance is the use of the relative angular velocity (rather than Euler angle rates) for the generalized velocities vector for a free–flying system. The overall number of degrees of freedom for the manipulator is given by $\mathcal{N} = \sum_{k=1}^{n} r(k)$.

The spatial velocity, $V(k)$, of the k^{th} body frame $\mathcal{O}_k$ is defined as $V(k) = \begin{pmatrix} \omega(k) \\ v(k) \end{pmatrix} \in \Re^6$, with $\omega(k)$ and $v(k)$ denoting the angular and linear velocities of $\mathcal{O}_k$. The relative spatial velocity across the k^{th} hinge is given by $H^*(k)\beta(k)$ where $H^*(k) \in \Re^{6\times r(k)}$ is the joint map matrix for the hinge. The spatial force of interaction, $f(k)$, across the k^{th} hinge is denoted $f(k) = \begin{pmatrix} N(k) \\ F(k) \end{pmatrix} \in \Re^6$, with $N(k)$ and $F(k)$ denoting the moment and force components respectively. The spatial inertia $M(k)$ of the k^{th} link referred to $\mathcal{O}_k$ is defined as

$$M(k) = \begin{pmatrix} \mathcal{J}(k) & m(k)\tilde{p}(k) \\ -m(k)\tilde{p}(k) & m(k)I_3 \end{pmatrix} \in \Re^{6\times 6}$$

where $m(k)$ is the mass, $p(k) \in \Re^3$ is the vector from $\mathcal{O}_k$ to the center of mass, and $\mathcal{J}(k) \in \Re^{3\times 3}$ is the inertia of the k^{th} link about $\mathcal{O}_k$.

With $V(k)$ denoting the spatial velocity, $\alpha(k)$ the spatial acceleration, $f(k)$ the spatial force and $T(k)$ the generalized hinge force at $\mathcal{O}_k$ for the k^{th} link, the following Newton–Euler recursive equations [5,7] describe the equations of motion for the serial manipulator:

$$\left\{\begin{array}{rcl} & & V(n+1) = 0, \quad \alpha(n+1) = 0 \\ \textbf{for } \boldsymbol{k} & = & \boldsymbol{n \cdots 1} \\ V(k) & = & \phi^*(k+1,k)V(k+1) + H^*(k)\beta(k) \\ \alpha(k) & = & \phi^*(k+1,k)\alpha(k+1) + H^*(k)\dot{\beta}(k) + a(k) \\ \textbf{end loop} & & \end{array}\right.$$

$$\left\{\begin{array}{rcl} & & f(0) = 0 \\ \textbf{for } \boldsymbol{k} & = & \boldsymbol{1 \cdots n} \\ f(k) & = & \phi(k+1,k)f(k-1) + M(k)\alpha(k) + b(k) \\ T(k) & = & H(k)f(k) \\ \textbf{end loop} & & \end{array}\right. \tag{3.1}$$

Here $a(k)$ and $b(k)$ denote the velocity dependent nonlinear Coriolis acceleration and gyroscopic force terms respectively. $\phi(k, k-1)$ denotes the transformation operator between the $\mathcal{O}_{k-1}$ and $\mathcal{O}_k$ frames and is given by

$$\phi(k, k-1) = \begin{pmatrix} I_3 & \tilde{l}(k, k-1) \\ 0 & I_3 \end{pmatrix} \in \Re^{6\times 6}$$

We use spatial operators [5] to obtain compact expressions for the equations of motion and other key dynamical quantities. The vector $\theta \stackrel{\triangle}{=} [\theta^*(1), \cdots \theta^*(n)]^* \in \Re^{\mathcal{N}}$ denotes the vector of generalized coordinates for the manipulator. Similarly, we define the vectors of generalized velocities $\beta \in \Re^{\mathcal{N}}$ and generalized (hinge) forces $T \in \Re^{\mathcal{N}}$ for the manipulator. The vector of spatial velocities V is defined as $V \stackrel{\triangle}{=} [V^*(1) \cdots V^*(n)]^* \in \Re^{6n}$. The vector of spatial accelerations is denoted $\alpha \in \Re^{6n}$, that of the Coriolis accelerations by $a \in \Re^{6n}$ and of the link gyroscopic forces by $b \in \Re^{6n}$, and the link interaction spatial forces by $f \in \Re^{6n}$. The equations of motion for the serial manipulator can be written as follows (see reference [5] for details):

$$V = \phi^* H^* \beta \tag{3.2a}$$
$$\alpha = \phi^*[H^*\dot{\beta} + a] \tag{3.2b}$$
$$f = \phi[M\alpha + b] \tag{3.2c}$$
$$T = Hf = \mathcal{M}\dot{\beta} + \mathcal{C} \tag{3.2d}$$

where

$$\mathcal{M} \stackrel{\triangle}{=} H\phi M\phi^* H^* \in \Re^{\mathcal{N}\times\mathcal{N}} \tag{3.3a}$$
$$\mathcal{C} \stackrel{\triangle}{=} H\phi[M\phi^* a + b] \in \Re^{\mathcal{N}} \tag{3.3b}$$

and $H \stackrel{\triangle}{=} \text{diag}\left\{H(k)\right\}$, $M \stackrel{\triangle}{=} \text{diag}\left\{M(k)\right\}$,

$$\mathcal{E}_\phi \stackrel{\triangle}{=} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \phi(2,1) & 0 & \dots & 0 & 0 \\ 0 & \phi(3,2) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \phi(n,n-1) & 0 \end{pmatrix} \tag{3.4a}$$

$$\phi \stackrel{\triangle}{=} (I - \mathcal{E}_\phi)^{-1} = \begin{pmatrix} I & 0 & \dots & 0 \\ \phi(2,1) & I & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \phi(n,1) & \phi(n,2) & \dots & I \end{pmatrix} \tag{3.4b}$$

with

$$\phi(i,j) \stackrel{\triangle}{=} \phi(i,i-1) \ \cdots \ \phi(j+1,j) \ \text{ for } \ i > j$$

$\mathcal{M}$ is the *mass matrix* for the manipulator and the vector $\mathcal{C}$ contains the velocity dependent Coriolis and gyroscopic hinge forces. External forces on a link are handled by adding their sum effect to the component of the b vector for that link.

Using the index 0 to represent the end–effector frame, the spatial velocity, $V(0)$, of the end–effector is given by

$$V(0) = \phi^*(1,0)V(1) = BV = B\phi^* H^* \dot{\beta} \tag{3.5}$$

where the operator B is

$$B \stackrel{\triangle}{=} [\phi^*(1,0), 0, \cdots 0]^* \in \Re^{6n \times 6}$$

From (3.5) it follows that the operator expression for the end–effector Jacobian, J, is given by

$$J \stackrel{\triangle}{=} B^* \phi^* H^* \tag{3.6}$$

From here on, we extend the terminology *hinge* to include manipulator degrees of freedom that do not necessarily arise from physical hinges. This is possible because with spatial operators, both hinge and non–hinge manipulator degrees of freedom share similar mathematical representations. The operator formulation described in this section extends to manipulators such as free–flying space robots and flexible link manipulators which have degrees of freedom not associated with physical hinges. Such an extension of the operator formulation to flexible link manipulators is described in reference [8]. Only the detailed structure of the $\phi(.,.)$, $H(.)$ and $M(.)$ elements require modification. For free–flying space manipulators, the degrees of freedom associated with overall motion in free–space is modeled by attaching a 6 degree of freedom hinge between the base–body and the inertial frame.

4 Modeling of Under–Actuated Manipulators

We now turn to the topic of under–actuated manipulators, i.e., manipulators with more degrees of freedom than control actuators. As mentioned earlier, we use the term *active* degree of freedom for a manipulator degree of freedom associated with a control actuator. Conversely, a *passive* degree of freedom is a manipulator degree of freedom with no control actuator. Due to the presence of friction, stiffness etc., the generalized force associated with a passive degree of freedom need not necessarily be zero. For a free–flying space manipulator all the manipulator internal hinge degrees of freedom are active degrees of freedom. However, the six positional and orientation degrees of freedom for the manipulator as a whole represent passive degrees of freedom. In the case of manipulators with link or joint flexibility, the degrees of freedom associated with the link deformation are passive, while the hinge degrees of freedom are all active. In this instance, the generalized forces for the passive degrees of freedom are non–zero and contain contributions from the elastic stiffness and damping forces.

Typically, the component degrees of freedom of a multiple degree of freedom hinge are either all active or all passive. In the former case the hinge is denoted an *active hinge* and in the latter, a *passive hinge*. In certain situations, such as due to actuator failures, it is possible to have multiple degree of freedom hinges with a mix of active and passive component degrees of freedom. However, for modeling purposes, such multiple degree of freedom hinges can be decomposed into equivalent concatenation of active and passive hinges. In the rest of the discussion, we assume that a manipulator model with hinges containing a mix of passive and active component degrees of freedom has been replaced by an equivalent manipulator model containing only active and passive hinges.

The number of passive hinges in the manipulator is denoted n_p, and $\mathcal{I}_p$ denotes the set of their indices. $\mathcal{I}_a$ denotes the corresponding set of indices of the active hinges and $n_a = (n - n_p)$ is the number of active hinges in the manipulator. The total number of passive degrees of freedom is given by $\mathcal{N}_p$ $(= \sum_{k \in \mathcal{I}_p} r(k))$, while the total number of active degrees of freedom is given by $\mathcal{N}_a$ $(= \sum_{k \in \mathcal{I}_a} r(k))$. Note that $\mathcal{N}_a + \mathcal{N}_p = \mathcal{N}$.

We use the sets of hinge indices, $\mathcal{I}_a$ and $\mathcal{I}_p$, to decompose the manipulator into a pair of manipulator subsystems: the *active arm* $\mathcal{A}_a$, and the *passive arm* $\mathcal{A}_p$. $\mathcal{A}_a$ is the $\mathcal{N}_a$ degree of freedom manipulator resulting from freezing all the passive hinges (i.e. all hinges whose index is in $\mathcal{I}_p$), while $\mathcal{A}_p$ is the $\mathcal{N}_p$ degree of freedom manipulator resulting from freezing all the active hinges (i.e. all hinges whose index is in the set $\mathcal{I}_a$). This decomposition is illustrated in Figure 4.

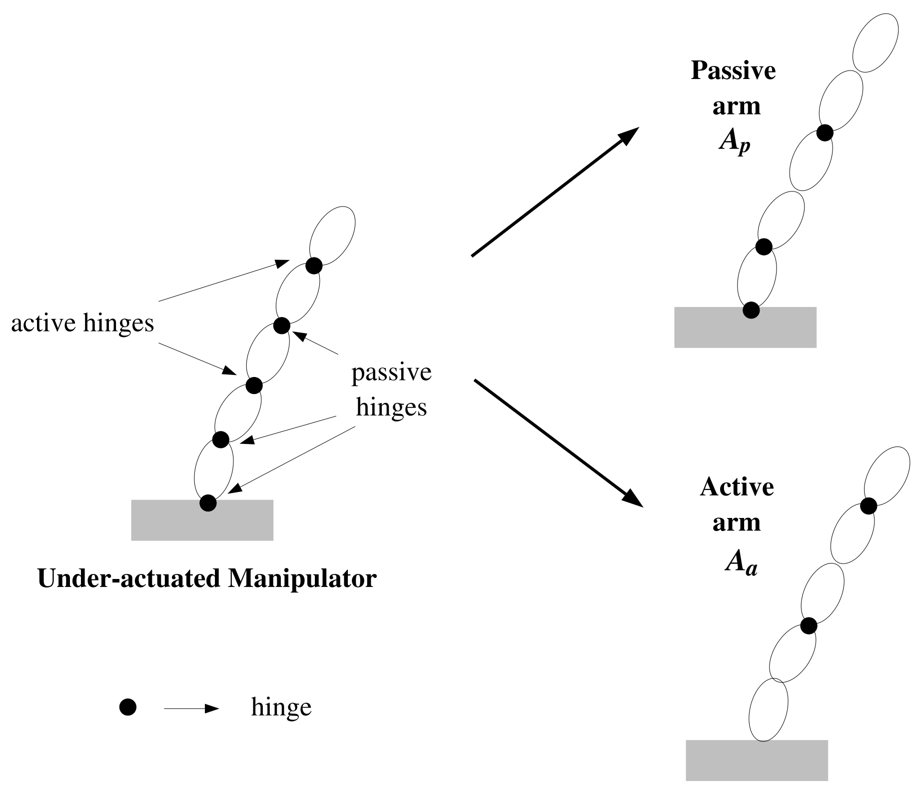

Figure 2: Decomposition of an under–actuated manipulator into component active and passive arms

Let $\beta_a \in \Re^{\mathcal{N}_a}$, $T_a \in \Re^{\mathcal{N}_a}$ and $H_a^* \in \Re^{6n \times \mathcal{N}_a}$ denote the vector of generalized velocities, the vector of hinge forces and the joint map matrix for arm $\mathcal{A}_a$. Similarly, let $\beta_p \in \Re^{\mathcal{N}_p}$, $T_p \in \Re^{\mathcal{N}_p}$ and $H_p^* \in \Re^{6n \times \mathcal{N}_p}$ denote the corresponding quantities for arm $\mathcal{A}_p$. The two vectors $\beta_a \in \Re^{\mathcal{N}_a}$ and $\beta_p \in \Re^{\mathcal{N}_p}$, also represent a decomposition of the vector of generalized velocities β in a manner consistent with the sets $\mathcal{I}_a$ and $\mathcal{I}_p$ respectively. Similarly T_a and T_p are decompositions of T, and H_a^* and H_p^* are decompositions of H^*. The columns of H^* that correspond to the active hinges appear as columns of H_a^*, while those that correspond to the passive hinges appear as columns of H_p^*. Thus it follows that

$$H_a^* \beta_a + H_p^* \beta_p = H^* \beta \tag{4.1}$$

4.1 Equations of Motion

We use the above decomposition to rewrite the equations of motion of (3.2) in the following partitioned form:

$$\begin{pmatrix} \mathcal{M}_{aa} & \mathcal{M}_{ap} \\ \mathcal{M}_{ap}^* & \mathcal{M}_{pp} \end{pmatrix} \begin{pmatrix} \dot{\beta}_a \\ \dot{\beta}_p \end{pmatrix} + \begin{pmatrix} \mathcal{C}_a \\ \mathcal{C}_p \end{pmatrix} = \begin{pmatrix} T_a \\ T_p \end{pmatrix} \tag{4.2}$$

where, with $i, j \in \{p, a\}$ we have

$$\mathcal{M}_{ij} \stackrel{\triangle}{=} H_i \phi M \phi^* H_j^*, \text{ and } \mathcal{C}_i \stackrel{\triangle}{=} H_i \phi [b + M \phi^* a] \tag{4.3}$$

In (4.2), the submatrices $\mathcal{M}_{aa}$ and $\mathcal{M}_{pp}$ are the mass matrices for the $\mathcal{A}_a$ and $\mathcal{A}_p$ arms respectively.

For manipulator control, we need to compute the actuator forces required to obtain a desired motion of the active hinges and the resulting motion induced at the passive hinges. That is, it is necessary to compute the active hinge forces T_a required to obtain a desired active hinge acceleration $\dot{\beta}_a$, and the resulting acceleration $\dot{\beta}_p$ induced at the passive hinges. We assume here that models for the passive hinges are available and can be used to compute the passive hinge forces vector T_p. These models will typically account for effects such as friction, backlash, stiffness etc. A simple rearrangement of (4.2) puts it in the form

$$\begin{pmatrix} T_a \\ \dot{\beta}_p \end{pmatrix} = \begin{pmatrix} \mathcal{S}_{aa} & \mathcal{S}_{ap} \\ -\mathcal{S}_{ap}^* & \mathcal{S}_{pp} \end{pmatrix} \begin{pmatrix} \dot{\beta}_a \\ T_p \end{pmatrix} + \begin{pmatrix} \mathcal{C}_a - \mathcal{S}_{ap}\mathcal{C}_p \\ -\mathcal{S}_{pp}\mathcal{C}_p \end{pmatrix} \tag{4.4}$$

where

$$\begin{aligned} \mathcal{S}_{aa} &\stackrel{\triangle}{=} \mathcal{M}_{aa} - \mathcal{M}_{ap}\mathcal{M}_{pp}^{-1}\mathcal{M}_{ap}^* \\ \mathcal{S}_{ap} &\stackrel{\triangle}{=} \mathcal{M}_{ap}\mathcal{M}_{pp}^{-1} \\ \mathcal{S}_{pp} &\stackrel{\triangle}{=} \mathcal{M}_{pp}^{-1} \end{aligned} \tag{4.5}$$

The direct use of the expression on the right of (4.4) to obtain $\dot{\beta}_p$ and T_a requires the computation of $\mathcal{M}$, the inversion of $\mathcal{M}_{pp}$ and the formation of various matrix/matrix and matrix/vector products. The computational cost of this dynamics algorithm is *cubic* in $\mathcal{N}_p$ and *quadratic* in $\mathcal{N}_a$. Later, in Section 5 we describe a new $O(\mathcal{N})$ dynamics algorithm that does not require the computation of $\mathcal{M}$, and whose computational complexity is only *linear* in both $\mathcal{N}_a$ and $\mathcal{N}_p$

4.2 Spatial Operator Expression for $\mathcal{M}_{pp}^{-1}$

Since $\mathcal{M}_{pp}$ is the mass matrix of the passive arm $\mathcal{A}_p$, operator factorization and inversion techniques developed for regular manipulator mass matrices can be used to obtain a closed form spatial operator expression for $\mathcal{M}_{pp}^{-1}$. In this section, we briefly describe the applicable results and refer the reader to reference [5] for additional details. First, we define the quantities $P(.), D_p(.), G_p(.), K_p(.), \overline{\tau}_p(.), P^+(.)$ and $\psi(.,.)$ for the manipulator links using the following recursive algorithm:

$$\left\{\begin{array}{l}
\quad P^+(0) = 0 \\
\textbf{for } \boldsymbol{k} = \boldsymbol{1 \cdots n} \\
\quad P(k) = \phi(k,k-1)P^+(k-1)\phi^*(k,k-1) + M(k) \\
\quad \left\{\begin{array}{l}
\textbf{if } k \in \mathcal{I}_p \\
\quad \overline{\tau}_p(k) = I_6 \\
\textbf{else} \\
\quad D_p(k) = H_p(k)P(k)H_p^*(k) \\
\quad G_p(k) = P(k)H_p^*(k)D_p^{-1}(k) \\
\quad K_p(k+1,k) = \phi(k+1,k)G_p(k) \\
\quad \overline{\tau}_p(k) = I - G_p(k)H_p(k) \\
\textbf{end if}
\end{array}\right. \\
\quad P^+(k) = \overline{\tau}_p(k)P(k) \\
\quad \psi(k+1,k) = \phi(k+1,k)\overline{\tau}_p(k) \\
\textbf{end loop}
\end{array}\right. \tag{4.6}$$

I_6 above denotes the 6×6 identity matrix. The quantities defined in (4.6) are very similar to the articulated body quantities required for the $O(\mathcal{N})$ forward dynamics algorithm for regular manipulators [5, 9]. When we restrict our attention to the hinges of the $\mathcal{A}_p$ (passive) arm alone, these quantities are precisely the articulated body quantities for the $\mathcal{A}_p$ manipulator. The recursion in (4.6) proceeds from the tip to the base of the manipulator. At each hinge, the active or passive status of the hinge is checked. Depending on the status of the hinge, the appropriate computations are carried out and the recursion proceeds to the next hinge. This continues until the base–body is reached.

The operator $P \in \Re^{6n \times 6n}$ is defined as a block diagonal matrix with its k^{th} diagonal element being $P(k) \in \Re^{6 \times 6}$. The quantities in (4.6) are also used to define the following spatial operators:

$$\begin{aligned}
D_p &\stackrel{\triangle}{=} H_p P H_p^* \in \Re^{\mathcal{N}_p \times \mathcal{N}_p} \\
G_p &\stackrel{\triangle}{=} P H_p^* D_p^{-1} \in \Re^{6n \times \mathcal{N}_p} \\
K_p &\stackrel{\triangle}{=} \mathcal{E}_\phi G_p \in \Re^{6n \times \mathcal{N}_p} \\
\overline{\tau}_p &\stackrel{\triangle}{=} I - G_p H_p \in \Re^{6n \times 6n} \\
\mathcal{E}_\psi &\stackrel{\triangle}{=} \mathcal{E}_\phi \overline{\tau}_p \in \Re^{6n \times 6n}
\end{aligned} \tag{4.7}$$

The operators D_p, G_p and $\overline{\tau}_p$ are all block diagonal. Even though K_p and $\mathcal{E}_\psi$ are not block diagonal matrices, their only nonzero block elements are the elements $K_p(k,k-1)$'s and $\psi(k,k-1)$'s respectively along the first subdiagonal. It is easy to verify from (4.6) that P satisfies the equation

$$M = P - \mathcal{E}_\psi P \mathcal{E}_\psi^* = P - \mathcal{E}_\phi P \mathcal{E}_\psi^* \tag{4.8}$$

Now define the lower–triangular operator $\psi \in \Re^{6n \times 6n}$ as

$$\psi \stackrel{\triangle}{=} (I - \mathcal{E}_\psi)^{-1} \tag{4.9}$$

Its block elements $\psi(i,j) \in \Re^{6 \times 6}$ are given by

$$\psi(i,j) \stackrel{\triangle}{=} \left\{\begin{array}{lll}
\psi(i,i-1) \; \cdots \; \psi(j+1,j) & \text{for} & i > j \\
I & \text{for} & i = j \\
0 & \text{for} & i < j
\end{array}\right.$$

The structure of the operators ψ and $\mathcal{E}_\psi$ is identical to that of the operators ϕ and $\mathcal{E}_\phi$ except that the elements are now $\psi(i,j)$ rather than $\phi(i,j)$.

We refer to the expression for $\mathcal{M}_{pp}$ in (4.3) as the Newton–Euler factorization of the passive arm mass matrix. We now use results from references [5, 6] to develop alternative operator factorization and inversion expressions. The proofs of Lemmas 4.1–4.3 can be found in [5].

Lemma 4.1 *The innovations factorization of the mass matrix $\mathcal{M}_{pp}$ is given by*

$$\mathcal{M}_{pp} = [I + H_p\phi K_p]D_p[I + H_p\phi K_p]^* \tag{4.10}$$

∎

The factorization in Lemma 4.1 can also be regarded as a closed–form LDL^* factorization of $\mathcal{M}_{pp}$. The closed form operator expression for the inverse of the factor $[I + H_p\phi K_p]$ is described in Lemma 4.2 below.

Lemma 4.2 $[I + H_p\phi K_p]^{-1} = [I - H_p\psi K_p]$ ∎

Combining Lemma 4.1 and Lemma 4.2 leads to the following closed form operator expression for the inverse of the mass matrix $\mathcal{M}_{pp}$.

Lemma 4.3 $\mathcal{M}_{pp}^{-1} = [I - H_p\psi K_p]^* D_p^{-1}[I - H_p\psi K_p]$ ∎

The factorization in Lemma 4.3 can be regarded as a closed–form LDL^* factorization of $\mathcal{M}_{pp}^{-1}$.

5 Recursive $O(\mathcal{N})$ Dynamics Algorithm

The following lemma uses the operator expression for the inverse of $\mathcal{M}_{pp}$ together with (4.5), to derive new closed form operator expressions for the $\mathcal{S}_{ij}$ matrices.

Lemma 5.1

$$\begin{aligned}
\mathcal{S}_{pp} &= [I - H_p\psi K_p]^* D_p^{-1}[I - H_p\psi K_p] && (5.1a)\\
\mathcal{S}_{ap} &= H_a\left\{\psi K_p + P\psi^* H_p^* D_p^{-1}[I - H_p\psi K_p]\right\} &&\\
&= H_a\left\{(\psi - P\Omega)K_p + P\psi^* H_p^* D_p^{-1}\right\} && (5.1b)\\
\mathcal{S}_{aa} &= H_a[\psi M\psi^* - P\Omega P]H_a^* = H_a[(\psi - P\Omega)P + P\tilde{\psi}^*]H_a^* && (5.1c)
\end{aligned}$$

where

$$\Omega \stackrel{\triangle}{=} \psi^* H_p^* D_p^{-1} H_p\psi \tag{5.2}$$

Proof: See Appendix A. ∎

The expressions for the $\mathcal{S}_{ij}$ matrices in Lemma 5.1 require only the inverse of the block–diagonal matrix D_p — an inverse that is relatively easy to obtain.

We now derive a recursive $O(\mathcal{N})$ algorithm for the dynamics of under–actuated manipulators. One of the primary computations for manipulator control is the computation of the actuator generalized forces

T_a needed to obtain a desired acceleration $\dot{\beta}_a$ at the active hinges and the resulting acceleration $\dot{\beta}_p$ induced at the passive hinges. We use Lemma 5.1 to obtain expressions for $\dot{\beta}_p$, the active hinge forces vector T_a, and the link spatial accelerations vector α and express them more simplify using the new quantities z, ϵ_p and ν_p defined below.

Lemma 5.2

$$\begin{aligned}
\alpha &= \psi^* \left[H_p^* \nu_p + H_a^* \dot{\beta}_a + a \right] && (5.3a) \\
\dot{\beta}_p &= [I - H_p \psi K_p]^* \nu_p - K_p^* \psi^* [H_a^* \dot{\beta}_a + a] \\
&= \nu_p - K_p^* \alpha && (5.3b) \\
T_a &= H_a \left\{ z + P[\psi^* H_p^* \nu_p + \tilde{\psi}^* (H_a^* \dot{\beta}_a + a)] \right\} \\
&= H_a P[\alpha - H_a^* \dot{\beta}_a - a] + H_a z && (5.3c)
\end{aligned}$$

where

$$\begin{aligned}
z &\stackrel{\triangle}{=} \psi \left[K_p T_p + b + P(H_a^* \dot{\beta}_a + a) \right] \\
\epsilon_p &\stackrel{\triangle}{=} T_p - H_p z \\
\nu_p &\stackrel{\triangle}{=} D_p^{-1} \epsilon_p
\end{aligned}$$

Proof: See Appendix A. ∎

The ability to convert spatial operator expressions into fast recursive algorithms by inspection is one of the advantages of the spatial operator approach. This is a direct consequence of the special structure of operators such as ϕ and ψ. We use this feature to convert the closed form operator expressions for the vectors $\dot{\beta}_p$ and T_a in Lemma 5.2 into a recursive $O(\mathcal{N})$ computational algorithm. This algorithm requires a recursive tip–to–base sweep followed by a base–to–tip sweep as described below:

$$\left\{ \begin{array}{l}
\quad z(0) = 0 \\
\textbf{for } \boldsymbol{k} = \mathbf{1} \cdots \boldsymbol{n} \\
\quad \left\{ \begin{array}{lcl}
\textbf{if } k \in \mathcal{I}_a & & \\
\quad z(k) &=& \phi(k,k-1) z^+(k-1) + b(k) + P(k)[H^*(k) \dot{\beta}_a(k) + a(k)] \\
\quad z^+(k) &=& z(k) \\
\textbf{else} & & \\
\quad z(k) &=& \phi(k,k-1) z^+(k-1) + b(k) + P(k) a(k) \\
\quad \epsilon_p(k) &=& T_p(k) - H(k) z(k) \\
\quad z^+(k) &=& z(k) + G_p(k) \epsilon_p(k) \\
\quad \nu_p(k) &=& D_p^{-1} \epsilon_p(k) \\
\textbf{end if} & &
\end{array} \right. \\
\textbf{end loop}
\end{array} \right. \qquad (5.4a)$$

$$\left\{ \begin{array}{l}
\quad \alpha^+(n+1) = 0 \\
\textbf{for } \boldsymbol{k} = \boldsymbol{n} \cdots \mathbf{1} \\
\quad \alpha^+(k) = \phi^*(k+1,k) \alpha(k+1) \\
\quad \left\{ \begin{array}{lcl}
\textbf{if } k \in \mathcal{I}_a & & \\
\quad f(k) &=& P(k) \alpha^+(k) + z(k) \\
\quad T_a(k) &=& H(k) f(k) \\
\textbf{else} & & \\
\quad \dot{\beta}_p(k) &=& \nu_p(k) - G_p^*(k) \alpha^+(k) \\
\textbf{end if} & &
\end{array} \right. \\
\quad \alpha(k) = \alpha^+(k) + H^*(k) \dot{\beta}(k) + a(k) \\
\textbf{end loop}
\end{array} \right. \qquad (5.4b)$$

The recursion in (5.4a) starts from the tip of the manipulator and proceeds towards the base. At each hinge, the active/passive status of the hinge is checked. If the hinge is active, its acceleration is known and is used to update the residual force $z(\cdot)$. On the other hand, if the hinge is passive, its generalized force is known and is used to update the residual force. The recursion continues until the base is reached. Now begins the recursion in (5.4b) from the manipulator base towards the tip. This time, as each new hinge is encountered, its hinge acceleration is computed if it is a passive hinge, or else, its unknown generalized force is computed if it is an active hinge. This continues until the tip is reached and all the hinges have been processed. In summary, this dynamics algorithm requires the following 3 steps:

1. The recursive computation of all the link velocities $V(k)$, and the Coriolis terms $a(k)$ and $b(k)$ using a base-to-tip recursion sweep as in the standard Newton–Euler inverse dynamics algorithm in (3.1).
2. Recursive computation of the articulated body quantities using the tip-to-base recursive sweep described in (4.6).
3. The inward tip-to-base recursive sweep in (5.4a) to compute the residual forces $z(k)$. This is followed by the base-to-tip recursive sweep in (5.4b) to compute the components of $\dot{\beta}_p$, T_a and α.

Note that the recursions in Step (2) can be combined and carried out in conjunction with the tip-to-base sweep in Step (3).

We can regard this algorithm as a *generalized dynamics algorithm* for manipulators. An interesting feature of this algorithm is that its structure is a hybrid of known inverse and forward dynamics algorithms for regular manipulators. When all the hinges are passive, $\mathcal{I}_a$ is empty and the steps in the above algorithm reduce to the well known $O(\mathcal{N})$ articulated body forward dynamics algorithm [5,9] for regular manipulators. In this case, $P(k)$ is the articulated body inertia of all the links outboard of the k^{th} link. In the other extreme case, when all the hinges are active hinges, $\mathcal{I}_p$ is empty, and the steps in the algorithm reduce to the composite rigid body inertias based $O(\mathcal{N})$ inverse dynamics algorithm for regular manipulators [6]. In this case, $P(k)$ is the composite rigid body inertia of all the links that are outboard of the k^{th} link. For a general under–actuated manipulator with both passive and active hinges, $P(k)$ is formed by a combination of articulated and composite body inertia type computations for the links outboard of the k^{th} hinge. It is in fact the articulated body inertia for all the links outboard of the k^{th} link for the passive arm $\mathcal{A}_p$.

Since each recursive step in the above algorithm has a fixed computational cost per degree of freedom, the overall computational cost of the algorithm is *linear* in both $\mathcal{N}_a$ and $\mathcal{N}_p$, i.e. *linear* in $\mathcal{N}$. That is, this is an $O(\mathcal{N})$ dynamics algorithm. The computational cost per passive degree of freedom is larger than the corresponding cost for an active degree of freedom. Non–zero generalized forces at the passive hinges are accounted for in a very natural manner in the algorithm. Also, the overhead associated with transitions between passive and active status of the hinge is small. When such a transition occurs during run–time, the only change required is to update the sets $\mathcal{I}_p$ and $\mathcal{I}_a$.

6 Kinematical Quantities

The end–effector Jacobian matrix is widely used in motion planning and control of regular manipulators. This Jacobian characterizes the relationship between the incremental motion of the controlled hinge degrees of freedom and the incremental motion of the end–effector. In this section we define similar Jacobian–like quantities for under–actuated manipulators. Some of the new issues that arise in dealing with under–actuated manipulators are: (a) the incremental motion relationships must be defined in the acceleration rather than in the velocity domain; (b) the Jacobian–like quantities depend not only on the kinematical properties of the links but also on the inertial properties of the links; and (c) in addition to the motion of the end–effector there is also "disturbance" motion induced at the passive hinges by the motion of the active degrees of freedom. These new features are somewhat simpler in the special case of free–flying space manipulators with inactive base–body control because, as discussed in more detail in Section 7, these manipulators possess linear and angular momentum integrals of motion We briefly look at their properties here since they provide a convenient conceptual bridge between the fairly well understood properties of regular manipulators and those of general under–actuated manipulators.

For regular manipulators, the end–effector Jacobian matrix – denoted J – describes the velocity domain relationship between the incremental motion of the controlled (i.e. all the hinge) degrees of freedom and the incremental motion of the end–effector frame as follows:

$$V(0) = J\beta_a \tag{6.1}$$

Here $V(0)$ denotes the spatial velocity of the end–effector and β_a is the same as β since regular manipulators do not have any passive hinges. The Jacobian J is independent of dynamical quantities such as link masses and inertias and depends only upon their kinematical properties. Efficient recursive algorithms have been developed for Jacobian–related computations for regular manipulators.

(6.1) still holds for free–flying space manipulators with inactive base–body control and zero spatial momentum, and describes the motion induced at the end–effector due to the motion of the active degrees of freedom. However, the end–effector Jacobian J depends upon the kinematical as well as the inertial properties of the links. This Jacobian is also referred to as the *generalized Jacobian* [10, 11] and denoted by the symbol J_G. In addition to (6.1), there is an additional manipulator Jacobian, the *disturbance Jacobian*, J_D, which describes "disturbance" motion induced in the passive degrees of freedom (the base–body degrees of freedom) by the motion of the active hinges [10]. This additional relationship can be written in the velocity domain as

$$\beta_p = J_D\beta_a \tag{6.2}$$

The disturbance Jacobian is not meaningful for regular manipulators since these manipulators have no passive degrees of freedom. Like J_G, the disturbance Jacobian J_D also depends on both the inertial and the kinematical properties of the links. When some of the base–body control forces are non–zero, or when the spatial momentum is non–zero, additional "drift" terms must be added to the right hand sides of (6.1) and (6.2) to account for the effect of these forces. Space manipulator control requires not only the control of the end–effector motion but also of the motion of the base–body. The properties of these pair of Jacobian matrices are fundamental to the development of good control algorithms for such manipulators. Singularity analysis of J_G is used to study the desirable and undesirable regions of the workspace [10]. The Jacobian J_G is also used for space manipulator control using methods such as *resolved rate control* [11]. In reference [12], J_D is used to for space robot control with the additional objective of minimizing the disturbance imparted to the base–body of the manipulator.

Unlike the regular manipulator Jacobian J, the Jacobians J_G and J_D are not true Jacobians, that is, they are not gradients of any vector valued functions. However, on the plus side, this terminology conveys the key idea that these matrices define the relationship between the incremental motions of the controlled hinges and the quantities being controlled.

For general under–actuated manipulators, relationships such as (6.1) and (6.2) cannot, in general, be expressed directly in the velocity domain, but can only be expressed in the acceleration domain as follows:

$$\alpha(0) = J_G\dot{\beta}_a + \text{non–acceleration dependent terms} \tag{6.3}$$

and

$$\dot{\beta}_p = J_D\dot{\beta}_a + \text{non–acceleration dependent terms} \tag{6.4}$$

The non–acceleration dependent terms on the right hand sides of (6.3) and (6.4) depend on the manipulator state and the passive hinge forces. The coefficient matrices J_G and J_D in (6.3) and (6.4) characterize (in the acceleration) domain the effect of the incremental motion of the controlled active hinges upon the incremental motion of the end–effector and the passive hinges respectively. Consequently, we adopt the terminology from the domain of space manipulators and continue to refer to these matrices as the generalized and disturbance Jacobians.

Later in this section, we derive expressions and computational algorithms for these Jacobians. We first define a pair of projection operators, $\mathcal{T}$ and $\overline{\mathcal{T}}$, as follows:

$$\mathcal{T} \stackrel{\triangle}{=} \Omega M, \quad \text{and} \quad \overline{\mathcal{T}} \stackrel{\triangle}{=} I - \mathcal{T} = I - \Omega M$$

where Ω is defined in (5.2). The following lemma shows that $\mathcal{T}$ and $\overline{\mathcal{T}}$ are indeed projection operators.

Lemma 6.1 *The operators $\mathcal{T}$ and $\overline{\mathcal{T}}$ are projection operators, i.e.,*

$$\mathcal{T}^2 = \mathcal{T}, \quad \text{and} \quad \overline{\mathcal{T}}^2 = \overline{\mathcal{T}} \tag{6.5}$$

Moreover, rank $[\mathcal{T}] = \mathcal{N}_p$, and rank$[\overline{\mathcal{T}}] = \mathcal{N}_a$. Also,

$$\overline{\mathcal{T}}\phi^* = \overline{\mathcal{T}}\psi^* = \psi^* - \Omega P \tag{6.6}$$

$$\overline{\mathcal{T}}\phi^* H_p^* = \overline{\mathcal{T}}\psi^* H_p^* = 0 \tag{6.7}$$

Proof: See Appendix A. ∎

From the expression for T_a in (4.4) it follows that $\mathcal{S}_{aa}$ is the mass matrix for the under–actuated manipulator. The projection operator $\overline{\mathcal{T}}$ provides a new expression for this mass matrix.

Lemma 6.2

$$\mathcal{S}_{aa} = H_a\phi(\overline{\mathcal{T}}^* M\overline{\mathcal{T}})\phi^* H_a^* = H\phi(\overline{\mathcal{T}}^* M\overline{\mathcal{T}})\phi^* H^* \tag{6.8}$$

Proof: See Appendix A. ∎

Note that the mass matrix of the regular manipulator is given by $H\phi M\phi^* H^*$, while that of the active arm $\mathcal{A}_a$ is given by $H_a\phi M\phi^* H_a^*$. Thus the mass matrix of the under–actuated manipulator is related to the mass matrices of the regular and active manipulators in a simple manner by the projection operator $\overline{\mathcal{T}}$.

6.1 The Generalized Jacobian J_G

The *generalized Jacobian* $J_G \in \Re^{6\times n_a}$ defines the relationship between the incremental motions of the active hinges and of the end–effector frame. Combining together the expressions in (5.3) it follows that the expression for the link spatial accelerations α is

$$\alpha = [\psi^* - \Omega P]H_a^*\dot{\beta}_a + \psi^* H_p^* D^{-1}[I - H_p\psi K_p]T_p - \Omega[b + Pa] + \psi^* a \tag{6.9}$$

Thus the spatial acceleration of the end–effector frame, $\alpha(0)$ is given by

$$\begin{aligned}
\alpha(0) &= B^*\alpha + a(0) \\
&= B^*[\psi^* - \Omega P]H_a^*\dot{\beta}_a + B^*\psi^* H_p^* D^{-1}[I - H_p\psi K_p]T_p - B^*\Omega[b + Pa] + B^*\psi^* a + a(0) \\
&= J_G\dot{\beta}_a + \text{velocity and } T_p \text{ dependent terms}
\end{aligned} \tag{6.10}$$

The expression for the generalized Jacobian J_G is given in the following lemma.

Lemma 6.3 *The generalized Jacobian J_G is given by*

$$J_G \triangleq B^*[\psi^* - \Omega P]H_a^* = B^*\overline{\mathcal{T}}\phi^* H_a^* \tag{6.11}$$

Proof: See Appendix A. ∎

It is clear from (6.11) that the kinematical as well as the inertial properties of the links enter into the structure of the Jacobian via the projection operator $\overline{\mathcal{T}}$. In contrast, for regular manipulators, the end–effector Jacobian is purely a function of the kinematical properties of the manipulator. Comparing with (3.6), we see that the deviation of J_G from the Jacobian of the regular manipulator, J, is given by the projection operator $\overline{\mathcal{T}}$.

The computation of J_G can be carried out recursively. First, all the hinge velocities are set to zero. This makes the nonlinear velocity dependent terms, $a(k)$ and $b(k)$, zero for all the links. Also, the passive hinge forces, T_p, are set to zero. Next, all the articulated body quantities are computed using the tip-to-base recursion in (4.6). The following procedure then leads to the computation of the k^{th} column of J_G:

1. Set the hinge accelerations as follows:

$$\dot{\beta}_a(i) = \begin{cases} 1 & \text{for } i = k \\ 0 & \text{for } i \neq k \end{cases}$$

2. Use the tip-to-base and base-to-tip recursions in (5.4) to compute the spatial acceleration $\alpha(1)$ of the outer–most link.
3. The k^{th} column of J_G is $\phi^*(1,0)\alpha(1)$.

Repeating this procedure for each of the n_a columns yields the complete generalized Jacobian matrix J_G. The computational cost of this algorithm is $\mathcal{O}(\mathcal{N}n_a)$. For a given n_a dimensional vector x, setting $\dot{\beta}_a = x$ and carrying out a single evaluation of Step (2) above results in the $O(\mathcal{N})$ computation of the matrix-vector product $J_G x$. While the structure of this algorithm is also recursive, as in the case of Jacobian computations for regular manipulators, Step (2) requires a tip-to-base recursion in addition to the base-to-tip recursion needed for regular manipulators.

6.2 Computation of Active Hinge Forces for a Desired End–Effector Trajectory

The generalized Jacobian can be used to compute the active hinge generalized forces $T_a(t)$ time profile required to achieve a desired end–effector time trajectory. The end–effector trajectory is defined by the time profile of the end–effector spatial acceleration $\alpha(0)$ (denoted $\alpha_0(t)$) over the time interval of interest. We assume that the state of the manipulator is known at the beginning, i.e., the configuration $\theta(t_0)$ and hinge velocities $\beta(t_0)$ are known at the initial time $t = t_0$ and that an integration time step Δt is being used. A brief sketch of the computational steps at time t is described below.

1. Compute $J_G(t)$.
2. Use (6.10) to compute $\dot{\beta}_a(t)$ via

$$\dot{\beta}_a(t) = J_G^{-1}(t)[\alpha_0(t) - \text{velocity and } T_p \text{ dependent terms}]$$

3. For this $\dot{\beta}_a(t)$, compute $T_a(t)$ and $\dot{\beta}_p(t)$ using the generalized dynamics algorithm in Section 5

4. Integrate the hinge accelerations and velocities to obtain the hinge velocities $\beta(t + \Delta t)$ and configurations $\theta(t + \Delta t)$ at time $t + \Delta t$. Go back to Step 1 and repeat the steps for time $(t + \Delta t)$.

This iterative procedure results in a time profile for the actuator forces $T_a(t)$ required to achieve the desired end–effector trajectory. It also computes the trajectory of the passive hinges for the whole time interval. For simplicity, we have assumed above that J_G is square and non–singular. When it is singular, or when it is non–square because there are redundant active hinge degrees of freedom available or only a subset of end–effector variables are specified (such as for pointing applications), this procedure can be modified to use methods such as pseudo–inverses and least–squares solutions in ways similar to those for regular redundant manipulators.

These methods easily apply when frames other than the end–effector frames are of interest. The only change needed is to the B operator so that the Jacobian to the new frame rather than J_G is used for the computations.

6.3 The Disturbance Jacobian J_D

In applications where a larger number of active degrees of freedom are available than are needed to meet the primary objective of end–effector motion control, the redundant active degrees of freedom can be used to meet other secondary objectives. These secondary objectives can include goals such as the optimization of the passive hinge motion to minimize disturbances. The *disturbance Jacobian* J_D characterizes the inertial coupling between the active and the passive hinges. It describes the incremental "disturbance" motion induced in the passive hinges due to the incremental motion of the active hinges. From (4.4), it follows that the passive hinge accelerations are given by

$$\begin{aligned} \dot{\beta}_p &= -\mathcal{S}^*_{ap}\dot{\beta}_a + \mathcal{S}_{pp}[T_p - \mathcal{C}_p] \\ &= -\left\{K^*_p\psi^* + [I - H_p\psi K_p]^* D_p^{-1} H_p\psi P\right\} H^*_a\dot{\beta}_a + [I - H_p\psi K_p]^* D_p^{-1}\left\{T_p - H_p\psi\left(K_pT_p + b + M\phi^* a\right)\right\} \\ &= J_D\dot{\beta}_a + \text{velocity and } T_p \text{ dependent terms} \end{aligned} \tag{6.12}$$

Lemma 6.4 *The operator expression for the disturbance Jacobian J_D is given by:*

$$J_D = -\mathcal{S}^*_{a_p} = -[I - H_p\psi K_p]^* D_p^{-1} H_p\psi P H^*_a - K^*_p\psi^* H^*_a \tag{6.13}$$

Proof: This follows from (5.1b). ∎

The computation of J_D can be carried out simultaneously with the computation of J_G using the algorithm described earlier in Section 61. The k^{th} column of J_D is simply the vector $\dot{\beta}_p$ computed during the steps for the computation of the k^{th} column of J_G. The computational cost of this algorithm is also $\mathcal{O}(\mathcal{N}n_a)$.

7 Application to Free–Flying Space Manipulators

Free–flying space manipulators are an important example of under–actuated manipulators. We look at some of their properties and discuss the application of the formulation and algorithms of this paper to these systems. The configuration considered consists of a manipulator mounted on a free–flying space vehicle. The space vehicle is controlled in six degrees of freedom by an attitude and translation control system. Control occurs in a coordinate system that moves with the trajectory of the space vehicle. The manipulator motion is controlled by actuators acting at the hinges of the manipulator. One of the critical tasks anticipated for free–flying manipulation is to perform a maneuver in which the manipulator has to move (to grasp a truss for example), while the attitude/translation control system prevents the spacecraft from moving too much. A certain amount of spacecraft motion might be tolerable, as long as this does not compromise safety and stability of the manipulator task.

Performing the manipulation maneuvers with the attitude and translation control system inactive most of the time can help conserve fuel and is referred to here as *reaction–mode control*. The control system turns on when the disturbance motions in the base–body exceed prescribed bounds. One of the desirable goals of space manipulator control is to plan and execute manipulator motions that minimize the activation of the control system in order to conserve fuel. The internal hinges of the space manipulator hinge represent the active degrees of freedom, while the six base–body degrees of freedom represent passive degrees of freedom. During reaction-mode control, the passive hinge forces are zero, i.e. $T_p = 0$. These forces are non–zero only when the space vehicle control system is on. The motion planning problem for space manipulators consists of computing active hinge forces to execute a desired end–effector trajectory while minimizing base–body motion. We assume that the manipulator has redundant degrees of freedom that can be used to minimize the motion of the base–body. This problem can be solved using the algorithm in Sections 5 and 6. At each control sample time, the following steps are executed:

1. The algorithms in Section 6 are used to recursively compute the generalized and disturbance Jacobians J_G and J_D and form the composite Jacobian-like quantity $\begin{pmatrix} J_G \\ J_D \end{pmatrix}$

2. The combination of (6.10) and (6.12) characterizes the effect of the active hinge accelerations $\dot{\beta}_a$ upon the end–effector acceleration $\alpha(0)$ and the passive hinge accelerations $\dot{\beta}_p$ as follows:
$$\begin{pmatrix} J_G \\ J_D \end{pmatrix} \dot{\beta}_a = \begin{pmatrix} \alpha(0) \\ \dot{\beta}_p \end{pmatrix} + \text{velocity and } T_p \text{ dependent terms} \tag{7.1}$$
(7.1) is solved for the active hinge acceleration $\dot{\beta}_p$ using the desired value for the end–effector acceleration and a value of zero for the acceleration at the passive hinges. The composite Jacobian matrix might not be square in most cases. When there are only a limited number of degrees of freedom, (7.1) can only be solved approximately and some performance will be lost. On the other hand, when there are sufficient redundant degrees of freedom, an infinite number of solutions will exist. In this case the solution can be chosen to meet additional performance objectives.

3. Next the active hinge accelerations are used in the generalized dynamics algorithm of Section 5 to compute the active hinge forces T_a

The spacecraft control forces T_p can be directly used in the steps of this algorithm at those instants in time when these forces are not zero. The above procedure is repeated at every control sample time instant throughout the time interval of interest.

This approach complements methods that involve the integrals of motion associated with space manipulators. These methods have been used extensively in other studies of space manipulators [10–13]. The six base–body degrees of freedom are *ignorable coordinates* [14] because the kinetic energy of the manipulator does not depend on the orientation or location of the manipulator in free–space. With the manipulator kinetic energy given by $\frac{1}{2}\beta^* \mathcal{M} \beta$, the subset of the Lagrangian equations of motion corresponding to the passive degrees of freedom, i.e. the lower half of (4.2) is:
$$\frac{\boldsymbol{d}[\mathcal{M}_{pa}\beta_a + \mathcal{M}_{pp}\beta_p]}{\boldsymbol{dt}} - \frac{1}{2}\nabla_{\theta_p}[\beta^* \mathcal{M} \beta] = T_p \tag{7.2}$$
Since $\mathcal{M}$, and consequently the kinetic energy do not depend on the base–body hinge generalized coordinates θ_p, we have
$$\nabla_{\theta_p}[\beta^* \mathcal{M} \beta] = 0$$

The left-hand side of (7.2) is therefore an exact differential and it can be rewritten in the form

$$\mathcal{M}_{pa}\beta_a + \mathcal{M}_{pp}\beta_p = [\mathcal{M}_{pa},\ \mathcal{M}_{pp}]\beta = \int_{t_0}^{t} T_p dt + \text{constant} \tag{7.3}$$

The left hand side of (7.3) is precisely the 6–vector spatial momentum[1] of the whole space manipulator at time t. The constant on the right hand side is the spatial momentum at time t_0, and the integral term reflects the rate of change of the momentum. (7.3) is equivalent to a time–varying, non–linear constraint on the generalized velocities of the system.

During reaction-mode control, $T_p = 0$, and therefore the left–hand side of (7.3) is constant, i.e. the linear and angular momentum of the manipulator are conserved, and are integrals of motion for the manipulator. The conservation of linear momentum is a holonomic constraint and implies that the center of mass of the manipulator remains stationary. On the other hand, the conservation of angular momentum represents a nonholonomic constraint. Methods using these constraints have been developed for analyzing the kinematics, dynamics and control of space manipulators. These methods have primarily focused on the case when the right side of (7.3) is zero, i.e. when the manipulator has zero spatial momentum and is undergoing reaction–mode control. The simple form of the generalized and disturbance Jacobian relationships of (6.1) and (6.2) hold only for this special situation. The extensions proposed to handle the cases when either the spatial momentum is non–zero or when at least some of the base–body control forces are non–zero are non–trivial since the constraint equations are time-varying and are not as simple. While the study of the special nature of the constraints is important to gain insight into the control problem, the results of this paper provide good computational algorithms to support these methods.

8 Conclusions

The techniques developed in this paper are applicable to the general class of under–actuated manipulators. A number of important applications such as free–flying space robots, hyper–redundant manipulators, manipulators with structural flexibility, manipulators (loosely) grasping an articulated object, and manipulators with actuator failures involve under–actuated manipulators.

For analysis, under–actuated manipulators are decomposed into component active and passive manipulators. This decomposition is used to express the equations of motion in a partitioned form. Spatial operator techniques are used to simplify and develop closed–form expressions for the equations of motion. A new efficient and recursive $O(\mathcal{N})$ dynamics algorithm, whose complexity depends only linearly on the number of degrees of freedom has been described. This algorithm may be viewed as a generalized dynamics algorithm for manipulators. The structure of this algorithm is a hybrid combination of the well known inverse and forward dynamics algorithms for regular manipulators. It reduces to known inverse and forward dynamics algorithms when all the hinges are set to either all active or all passive status respectively. We also develop operator expressions and computational algorithms for the generalized and disturbance Jacobians for under–actuated manipulators. These Jacobians are useful for end–effector motion control and path planning for under–actuated manipulators.

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[1]The spatial momentum vector consists of the concatenation of the vectors of angular and linear momentum respectively.

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Appendix A: Proofs of the Lemmas

We first establish the following useful identities.

Lemma Appendix A:.1 *The following spatial operator identities are used in the proofs of the lemmas in this paper:*

$$\begin{aligned}
\psi^{-1} &= \phi^{-1} + K_p H_p && \text{(A.1)}\\
\psi K_p H_p \phi &= \phi - \psi && \text{(A.2)}\\
[I - H_p \psi K_p] H_p \phi &= H_p \psi && \text{(A.3)}\\
\phi K_p [I - H_p \psi K_p] &= \psi K_p && \text{(A.4)}\\
\psi M \psi^* &= \tilde{\psi} P + P \psi && \text{(A.5)}\\
\phi M \psi^* &= \tilde{\phi} P + P \psi && \text{(A.6)}\\
\Omega M \phi^* &= \mathcal{T} \phi^* = \phi^* - (\psi^* - \Omega P) = \phi^* H_p^* [K_p^* (\psi^* - \Omega P) + D_p^{-1} H_p \psi P] && \text{(A.7)}\\
(\psi - P\Omega) M \phi^* &= (\psi - P\Omega) P + P \tilde{\psi}^* = \psi M \psi^* - P \Omega P && \text{(A.8)}\\
&= \phi - (\psi - P\Omega) && \text{(A.9)}
\end{aligned}$$

Proof: From (4.9), we have that

$$\psi^{-1} = I - \mathcal{E}_\psi = (I - \mathcal{E}_\phi) + \mathcal{E}_\phi G_p H_p = \phi^{-1} + K_p H_p$$

This identity immediately leads to the identities (A.1) and (A.2). The identities in (A.3) and (A.4) follow easily from (A.2).

Pre and post multiplying (4.8) from the left and the right by the operators ψ and ψ^* respectively leads to (A.5). On the other hand, pre and post multiplying (4.8) from the left and the right by the operators ϕ and ψ^* respectively leads to (A.6).

We have that

$$\begin{aligned}
\Omega M \phi^* &= \psi^* H_p^* D_p^{-1} H_p \psi M \phi^*\\
&= \psi^* H_p^* D_p^{-1} H_p [\psi P + P \tilde{\phi}^*] \quad \text{(using (A.6))}\\
&= \Omega P + \psi^* H_p^* K_p^* \phi^*\\
&= \phi^* - (\psi^* - \Omega P) \quad \text{(using (A.2))}
\end{aligned}$$

Also

$$\begin{aligned}
\phi^* H_p^* K_p^* (\psi^* - \Omega P) &= \phi^* H_p^* K_p^* \psi^* [I - H_p^* D_p^{-1} H_p \psi P]\\
&= (\phi^* - \psi^*)[I - H_p^* D_p^{-1} H_p \psi P] \quad \text{(using (A.2))}\\
&= \phi^* - (\psi^* - \Omega P) - \phi^* H_p^* D_p^{-1} H_p \psi P
\end{aligned}$$

This establishes (A.7).

$$\begin{aligned}
(\psi - P\Omega) M \phi^* &= \tilde{\psi} P + P \phi^* - P[\phi^* - \psi^* + \Omega P] \quad \text{(using (A.6) and (A.7))}\\
&= \psi M \psi^* - P \Omega P \quad \text{(using (A.5))}
\end{aligned}$$

This establishes (A.8). ∎

Proof of Lemma 5.1:
(5.1a) is merely a restatement of Lemma 4.3. With regards (5.1b):

$$\begin{aligned}
\mathcal{S}_{ap} &= \mathcal{M}_{ap}\mathcal{M}_{pp}^{-1} \quad \text{(using (4.10))} \\
&= H_a\phi M\phi^* H_p^*[I - H_p\psi K_p]^* D_p^{-1}[I - H_p\psi K_p] \\
&= H_a\phi M\psi^* H_p^* D_p^{-1}[I - H_p\psi K_p] \quad \text{(using (A.3))} \\
&= H_a\left\{\phi K_p + P\psi^* H_p^* D_p^{-1}\right\}[I - H_p\psi K_p] \quad \text{(using (A.6))} \\
&= H_a\left\{\psi K_p + P\psi^* H_p^* D_p^{-1}[I - H_p\psi K_p]\right\} \quad \text{(using (A.4))} \\
&= H_a\left\{(\psi - P\Omega)K_p + P\psi^* H_p^* D_p^{-1}\right\}
\end{aligned}$$

For (5.1c):

$$\begin{aligned}
\mathcal{S}_{aa} &= \mathcal{M}_{aa} - \mathcal{M}_{ap}\mathcal{M}_{pp}^{-1}\mathcal{M}_{ap}^* = \mathcal{M}_{aa} - \mathcal{S}_{ap}\mathcal{M}_{ap}^* \\
&= H_a\left\{\phi - \psi K_p H_p\phi - P\psi^* H^* D_p^{-1}[I - H_p\psi K_p]H_p\phi\right\} M\phi^* H_a^* \quad \text{(using (5.1b))} \\
&= H_a\left\{\psi - P\psi^* H^* D_p^{-1} H_p\psi\right\} M\phi^* H_a^* \quad \text{(using (A.2) and (A.3))} \\
&= H_a\left\{\psi - P\Omega\right\} M\phi^* H_a^* \\
&= H_a\left\{(\psi - P\Omega)P + P\tilde{\psi}^*\right\} H_a^* \quad \text{(using (A.8))}
\end{aligned}$$

■

Proof of Lemma 5.2:
From (4.4) we have that

$$\begin{aligned}
\dot{\beta}_p &= \mathcal{S}_{pp}[T_p - \mathcal{C}_p] - \mathcal{S}_{ap}^*\dot{\beta}_a \\
&= [I - H_p\psi K_p]^* D_p^{-1}\left\{T_p - H_p\psi\left(K_pT_p + b + M\phi^* a\right)\right\} - \\
&\qquad \left\{K_p^*\psi^* + [I - H_p\psi K_p]^* D_p^{-1} H_p\psi P\right\} H_a^*\dot{\beta}_a \quad \text{(using (A.3))} \\
&= [I - H_p\psi K_p]^* D_p^{-1}\left\{T_p - H_p\psi\left(K_pT_p + Pa + b\right)\right\} - K_p^*\psi^* a - \\
&\qquad \left\{K_p^*\psi^* + [I - H_p\psi K_p]^* D_p^{-1} H_p\psi P\right\} H_a^*\dot{\beta}_a \quad \text{(using (A.6) and (A.4))} \\
&= [I - H_p\psi K_p]^* D_p^{-1}\left\{T_p - H_p\psi\left(K_pT_p + P[H_a^*\dot{\beta}_a + a] + b\right)\right\} - K_p^*\psi^*[H_a^*\dot{\beta}_a + a] \\
&= [I - H_p\psi K_p]\nu_p - K_p^*\psi^*[H_a^*\dot{\beta}_a + a]
\end{aligned} \tag{A.10}$$

This establishes (5.3b). Also,

$$\begin{aligned}
\alpha &= \phi^*[H^*\dot{\beta} + a] = \phi^*[H_a^*\dot{\beta}_a + H_p^*\dot{\beta}_p + a] \\
&= \phi^* H_p^*\left\{[I - H_p\psi K_p]^*\nu_p - K_p^*\psi^*[H_a^*\dot{\beta}_a + a]\right\} + \phi^*[H_a^*\dot{\beta}_a + a] \quad \text{(using (A.10))} \\
&= \psi^*[H_p^*\nu_p + H_a^*\dot{\beta}_a + a] \quad \text{(using (A.2) and (A.3))}
\end{aligned}$$

This establishes (5.3a). From (A.7) it follows that

$$\mathcal{S}_{ap}\mathcal{C}_p = \mathcal{C}_a - H_a(\psi - P\Omega)[M\phi^* a + b] = \mathcal{C}_a - H_a(\psi - P\Omega)[Pa + b] - H_aP\tilde{\psi}^* a$$

Thus we have that

$$\begin{aligned}
T_a &= \mathcal{S}_{aa}\dot{\beta}_a + \mathcal{S}_{ap}[T_p - \mathcal{C}_p] + \mathcal{C}_a \\
&= H_a\left\{(\psi - P\Omega)P + P\tilde{\psi}^*\right\} H_a^*\dot{\beta}_a + H_a\left\{(\psi - P\Omega)K_p + P\psi^* H_a^* D_p^{-1}\right\} T_p \\
&\qquad + H_a(\psi - P\Omega)[Pa + b] + H_aP\tilde{\psi}^* a \\
&= H_a[I - P\psi^* H_p^* D_p^{-1} H_p]z + H_a[P\tilde{\psi}^*(H_a^*\dot{\beta}_a + a) + P\psi^* H_p^* D_p^{-1} T_p] \quad \text{(using (A.8))} \\
&= H_a\left\{z + P[\alpha - H_a^*\dot{\beta}_a - a]\right\}
\end{aligned}$$

This establishes (5.3c). ∎

Proof of Lemma 6.1:
It has been shown in reference [6] that

$$H_p \psi M \psi^* H_p^* = D_p \tag{A.11}$$

Thus

$$\Omega M \Omega = \psi^* H_p^* D_p^{-1} [H_p \psi M \psi^* H_p^*] D_p^{-1} H_p \psi = \Omega$$

This implies that not only $\mathcal{T}$, but also $\overline{\mathcal{T}} = I - \mathcal{T}$ are each projection operators. Since rank$[H_p] = \mathcal{N}_p$, and ψ and D_p are both full–rank operators, it implies that the rank of $\Omega = \psi^* H_p^* D_p^{-1} H_p \psi$ is $\mathcal{N}_p$. Since M is always full-rank, this implies that $\mathcal{T} = \Omega M$ is of rank $\mathcal{N}_p$. Since $\overline{\mathcal{T}} = I - \mathcal{T}$ is a projection operator, this implies that its rank is $\mathcal{N} - \mathcal{N}_p = \mathcal{N}_a$. (6.6) is simply a mild restatement of (A.7).

It is easy to verify that $\overline{\tau}_p P H_p^* = 0$, and so $\mathcal{E}_\psi P H_p^* = \tilde{\psi} P H_p^* = 0$. Since $\psi = I + \tilde{\psi}$, it follows that $\psi P H_p^* = P H_p^*$. Thus

$$\Omega P H_p^* = \psi^* H_p^* D_p^{-1} H_p P H_p^* = \psi^* H_p^* D_p^{-1} D_p = \psi^* H_p^*$$

Using this in (6.6) establishes (6.7). ∎

Proof of Lemma 6.2:
It is easy to verify that $\overline{\mathcal{T}}^* M \overline{\mathcal{T}} = M \overline{\mathcal{T}}$. Thus it follows that

$$\begin{aligned} \phi \overline{\mathcal{T}}^* M \overline{\mathcal{T}} \phi^* &= \phi M \overline{\mathcal{T}} \phi^* \\ &= \phi M \psi^* - \phi \overline{\mathcal{T}}^* P \quad \text{(using (6.6))} \\ &= (\psi - P\Omega)P + P\tilde{\psi}^* \quad \text{(using (A.8))} \end{aligned}$$

Pre- and post–multiplying the above with H_a and H_a^* and comparing with the expression for $\mathcal{S}_{aa}$ in (5.1c) establishes the lemma. ∎

Proof of Lemma 6.3:
The first equality follows from (6.10) and the second from (6.6). ∎